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Evolution Hemivariational Inequality for a Class of Dynamic Viscoelastic Nonmonotone Frictional Contact Problems

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Abstract—In this paper, we consider a class of hyperbolic hemivariational inequalities modeling the frictional contact between a viscoelastic body and a rigid foundation. The friction condition is described by a multivalued subdifferential relation which includes both the slip displacement and the slip rate dependent friction, and the Tresca models. The existence of weak solutions to the problem is proved by exploiting a surjectivity result for a class of pseudomonotone mappings.

Keywords—Hemivariational inequality, Friction, Viscoelasticity, Slip rate, Subdifferential, Contact problem, Nonconvex.

1. INTRODUCTION

Contact with friction is one of the oldest yet still poorly understood problems and dynamic friction has remained rarely explored until recently. Contact phenomena have received great importance in industrial mathematics and engineering sciences during last years largely due to considerable progress in mathematical modeling and experimental challenges. Dynamic frictional contact problems with viscoelastic materials have been considered only recently in a number of papers by Duvaut and Lions [1], Martins and Oden [2], Andrews *et al.* [3,4], Kuttler [5], Kuttler and Shillor [6,7], Awbi *et al.* [8], Ionescu and Sofonea [9], Han and Sofonea [10], and Shillor *et al.* [11]. For more literature, see also references in the last two monographs.

In this paper, we study the existence of solutions to the hemivariational inequality modeling the dynamic process of frictional contact between a deformable body and a foundation. The contact boundary condition is described by a general subdifferential relation for a locally Lipschitz function. A convex analysis approach to the contact problem under consideration is not possible and we are naturally lead to hemivariational inequality formulation. Hemivariational inequalities were introduced by Panagiotopoulos in [12,13] to describe various mechanical problems

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involving the nonconvex energy functionals. The derivation of hemivariational inequality is based on the notion of the generalized gradient of Clarke. In the case of convex superpotentials the hemivariational inequalities reduce to variational inequalities, see [1]. The existence results for hyperbolic hemivariational inequalities have been delivered only recently by Goeleven *et al.* [14], Haslinger *et al.* [15], and Migórski [16], who treated problems with a subdifferential depending on the first-order time derivative of the unknown function. Panagiotopoulos and Pop [17], Haslinger *et al.* [15], Ochal [18], and Migórski [19] have studied hemivariational inequalities with a multivalued term depending on the unknown function and not on its derivative. Frictional contact problems modeled by hemivariational inequalities have been considered in [14,16,20–23].

The present paper is a continuation of [22] and deals with a model for the dynamic problem of frictional contact. The viscoelastic body is supposed to satisfy the constitutive relationship of Kelvin-Voigt type with a linear elasticity operator and a nonlinear viscosity operator. We consider the time-dependent Tresca model assuming that the normal stress on the contact surface is prescribed. The friction is modeled by a subdifferential boundary condition which in particular covers the case when the tangential shear on the contact surface is a possibly multivalued function of the tangential velocity. The multivalued boundary condition has a nonmonotone character since this condition comes out from the nonconvex superpotential. The novelty here is that the friction subdifferential boundary condition is allowed to depend on both the displacement and the velocity. In particular, this means (see examples in Section 3) that in the contact problem the coefficient of friction is supposed to depend on the slip and/or slip rate. In a problem considered in [22] the friction coefficient depends only on the slip displacement. For related results on contact problems with the friction coefficient depending on the slip displacement, we refer to [24–26]. On the other hand the method used in this paper allows us to treat the multidimensional and multivalued friction laws of zig-zag type studied earlier by Panagiotopoulos [13].

In order to cope with the multivalued nonlinearities in the hemivariational inequality, we assume that the superpotential is regular in the sense of Clarke. This allows us to give an equivalent formulation of the hemivariational inequality under consideration in the form of evolution inclusion of second order. Next, we transform the latter, assuming a regular initial data, into an evolution inclusion of first order and apply a surjectivity result for L-pseudomonotone and coercive operators, cf. [27]. Then, by using a density argument we remove the restriction on the regularity of the initial condition and prove the result in a general case. We remark that our framework incorporates a nonlinear viscosity operator satisfying a pseudomonotonicity condition, a positive and in general noncoercive elasticity operator, and the subdifferential of a general nonsmooth superpotential.

The content of this paper is as follows. In Section 2 we recall some notation and present some auxiliary material. In Section 3 we state the mechanical problem, describe contact boundary conditions, and give some illustrative examples. We give the variational formulation of the model, derive a hemivariational inequality of evolution type, and state the hypotheses in Section 4. The main result on the existence of the weak solution to the hemivariational inequality as well as to the contact problem is delivered in Section 5. Section 6 contains proofs of some auxiliary results.

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2. PRELIMINARIES

In this section, we present the notation and recall some definitions needed in the sequel.

We denote by $\mathcal{S}_d$ the linear space of second-order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), or equivalently, the space $\mathbb{R}_s^{d\times d}$ of symmetric matrices of order d. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathcal{S}_d$ by

$$u \cdot v = u_i v_i, \qquad \|v\| = (v \cdot v)^{1/2}, \qquad \text{for all } u, v \in \mathbb{R}^d,$$
$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\|_{\mathcal{S}_d} = (\tau : \tau)^{1/2}, \qquad \text{for all } \sigma, \tau \in \mathcal{S}_d.$$

The summation convention over repeated indices is used.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let n denote the outward unit normal vector to Γ. The assumption that Γ is Lipschitz ensures that n is defined a.e. on Γ. We use the following spaces:

$$H = L^2(\Omega; \mathbb{R}^d), \qquad \mathcal{H} = \{\tau = \{\tau_{ij}\} : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega; \mathcal{S}_d),$$
$$H_1 = \{u \in H : \varepsilon(u) \in \mathcal{H}\} = H^1(\Omega; \mathbb{R}^d), \qquad \mathcal{H}_1 = \{\tau \in \mathcal{H} : \operatorname{div} \tau \in H\},$$

where ε and div denote the deformation and the divergence operators, respectively, given by

$$\varepsilon(u) = \{\varepsilon_{ij}(u)\}, \qquad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad \operatorname{div} \sigma = \{\sigma_{ij,j}\},$$

and the index following a comma indicates a partial derivative. The spaces H, $\mathcal{H}$, H_1, and $\mathcal{H}_1$ are Hilbert spaces equipped with the inner products

$$\langle u, v\rangle_H = \int_\Omega u \cdot v \, dx, \qquad \langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma : \tau \, dx,$$
$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \qquad \langle \sigma, \tau\rangle_{\mathcal{H}_1} = \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle \operatorname{div} \sigma, \operatorname{div} \tau\rangle_H.$$

The associated norms in H, $\mathcal{H}$, H_1, and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$, and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

For every $v \in H_1$ we denote by v its trace $\gamma_0 v$ on Γ, where $\gamma_0 : H^1(\Omega; \mathbb{R}^d) \to H^{1/2}(\Gamma; \mathbb{R}^d) \subset L^2(\Gamma; \mathbb{R}^d)$ is the trace map. Given $v \in H^{1/2}(\Gamma; \mathbb{R}^d)$ we denote by v_N and v_T the usual normal and the tangential components of v on the boundary Γ $v_N = v \cdot n$, $v_T = v - v_N n$. Similarly, for a regular tensor field $\sigma : \Omega \to \mathcal{S}_d$, we define its normal and tangential components by $\sigma_N = (\sigma n) \cdot n$ and $\sigma_T = \sigma n - \sigma_N n$. We also recall that the following Green formula holds: $\langle \sigma, \varepsilon(v)\rangle_{\mathcal{H}} + \langle \operatorname{div} \sigma, v\rangle_H = \int_\Gamma \sigma n \cdot v \, d\Gamma(x)$ for $v \in H_1$.

Given a reflexive Banach space Y, we denote by $\langle \cdot, \cdot \rangle_{Y^* \times Y}$ the duality pairing between Y and its dual Y^*. A single-valued operator $T : Y \to Y^*$ is said to be *pseudomonotone* if for each sequence $\{y_n\} \subseteq Y$ such that it converges weakly to $y_0 \in Y$ and $\limsup \langle T y_n, y_n - y_0\rangle_{Y^* \times Y} \le 0$, we have $\langle T y_0, y_0 - y\rangle_{Y^* \times Y} \le \liminf \langle T y_n, y_n - y\rangle_{Y^* \times Y}$ for all $y \in Y$.

We recall some definitions for a multivalued operator $T : Y \to 2^{Y^*}$ (see, e.g., [28] and [27]).

An operator T is said to be *pseudomonotone* if it satisfies

(a) for every $y \in Y$, Ty is a nonempty, convex, and weakly compact set in Y^*;
(b) T is upper semicontinuous from every finite-dimensional subspace of Y into Y^* endowed with the weak topology; and
(c) if $y_n \to y$ weakly in Y, $y_n^* \in T y_n$ and $\limsup \langle y_n^*, y_n - y\rangle_{Y^* \times Y} \le 0$, then for each $z \in Y$ there exists $y^*(z) \in Ty$ such that $\langle y^*(z), y - z\rangle_{Y^* \times Y} \le \liminf \langle y_n^*, y_n - z\rangle_{Y^* \times Y}$.

Let $L : D(L) \subset Y \to Y^*$ be a linear densely defined maximal monotone operator. An operator T is said to be *pseudomonotone with respect to* $D(L)$ (shortly L-*pseudomonotone*) if and only if (a) and (b) hold and

(d) if $\{y_n\} \subset D(L)$ is such that $y_n \to y$ weakly in Y, $L y_n \to L y$ weakly in Y^*, $y_n^* \in T(y_n)$, $y_n^* \to y^*$ weakly in Y^* and $\limsup \langle y_n^*, y_n\rangle_{Y^* \times Y} \le \langle y^*, y\rangle_{Y^* \times Y}$, then $(y, y^*) \in \operatorname{Graph}(T)$ and $\langle y_n^*, y_n\rangle_{Y^* \times Y} \to \langle y^*, y\rangle_{Y^* \times Y}$.

An operator T is said to be *coercive* if there exists a function $c : \mathbb{R}^+ \to \mathbb{R}$ with $c(r) \to \infty$ as $r \to \infty$ such that $\langle y^*, y\rangle_{Y^*\times Y} \geq c(\|y\|_Y)\|y\|_Y$ for every $(y, y^*) \in \mathrm{Graph}(T)$.

We recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $f : E \to \mathbb{R}$, where E is a Banach space (see [29]). The generalized directional derivative of f at $x \in E$ in the direction $v \in E$, denoted by $f^0(x; v)$, is defined by

$$f^0(x; v) = \limsup_{y\to x,\, \lambda\downarrow 0} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

The generalized gradient of f at x, denoted by $\partial f(x)$, is a subset of a dual space E^* given by $\partial f(x) = \{\zeta \in E^* : f^0(x; v) \geq \langle \zeta, v\rangle_{E^*\times E}$ for all $v \in E\}$. The locally Lipschitz function f is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $f'(x; v)$ exists and satisfies $f^0(x; v) = f'(x; v)$ for all $v \in E$. It is well known that a locally Lipschitz convex function is regular (cf. Proposition 2.3.6 of [29]).

Finally we state the properties of the generalized directional derivative and the generalized gradient which are needed in the sequel.

PROPOSITION 1. *Let X and Y be Banach spaces, $A \in \mathcal{L}(Y, X)$, and let $f : X \to \mathbb{R}$ be a locally Lipschitz function. Then*

(i) $(f \circ A)^0(x; z) \leq f^0(Ax; Az)$ *for* $x, z \in Y$,
(ii) $\partial (f \circ A)(x) \subseteq A^*\partial f(Ax)$ *for* $x \in Y$,

where $A^* \in \mathcal{L}(X^*, Y^*)$ denotes the adjoint operator to A. If f, $g : X \to \mathbb{R}$ are locally Lipschitz functions, then

(iii) $(f + g)^0(x; z) \leq f^0(x; z) + g^0(x; z)$ *for* $x, z \in X$,
(iv) $\partial (f + g)(x) \subseteq \partial f(x) + \partial g(x)$ *for* $x \in X$.

For the proof of the proposition compare Proposition 2.3.3, Theorem 2.3.10, and Remark 2.3.11 of [29].

3. PROBLEM STATEMENT

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain which represents a viscoelastic body with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. The boundary Γ is divided into three mutually disjoint measurable parts Γ_D, Γ_N, and Γ_C with $\mathrm{meas}(\Gamma_D) > 0$. The body is acted upon by time-dependent volume forces of density f_0 and surface tractions of density f_1 on Γ_N. We consider the process of evolution of the mechanical state on the time interval $[0, T]$. The body is clamped on Γ_D and may come in frictional contact along the surface Γ_C. We denote by $u = (u_1, \ldots, u_d)$ the displacement vector, by $\varepsilon(u) = (\varepsilon_{ij}(u))$ the linearized (small) strain tensor $\varepsilon_{ij}(u) = (1/2)(u_{i,j} + u_{j,i})$, and by $\sigma = (\sigma_{ij})$ the stress tensor, where $i, j = 1, \ldots, d$.

The mechanical problem consists of finding the displacement field $u : Q \to \mathbb{R}^d$ such that

$$\begin{aligned}
u''(t) - \operatorname{div} \sigma(t) &= f_0(t), && \text{in } Q, && (1)\\
\sigma(t) &= \mathcal{C}\varepsilon(u'(t)) + \mathcal{G}\varepsilon(u(t)), && \text{in } Q, && (2)\\
u(t) &= 0, && \text{on } \Gamma_D \times (0, T), && (3)\\
\sigma(t)n &= f_1(t), && \text{on } \Gamma_N \times (0, T), && (4)\\
-\sigma_N(t) &= S(t), && \text{on } \Gamma_C \times (0, T), && (5)\\
-\sigma_T(t) &\in \partial j(x, t, u(t), u'(t), u_T'(t)), && \text{on } \Gamma_C \times (0, T), && (6)\\
u(0) &= u_0, \quad u'(0) = u_1, && \text{in } \Omega, && (7)
\end{aligned}$$

where $Q = \Omega \times (0, T)$ while $\mathcal{C}$ and $\mathcal{G}$ are given nonlinear and linear constitutive functions, respectively.

In the model the equations of motion (1) are considered with the viscoelastic constitutive relationship of the Kelvin-Voigt type (2) while (3) and (4) represent the displacement and traction boundary conditions, respectively. Equation (5) states that the normal stress is prescribed on the boundary Γ_C and is given by $S \geq 0$. Such a condition makes sense when the real contact area is close to the nominal one and the surfaces are conforming. Then $S = S(x,t)$ is the contact pressure and it is given by the ratio of the total applied force to the total nominal contact area. It is considered (see Chapters 2.6 and 10.1 of [11]) to be a good approximation when the load is light and the contact force is transmitted by the asperity tips only. The functions u_0 and u_1 are the initial displacement and the initial velocity, respectively. In condition (6), the superpotential is allowed to depend separately on both the velocity u' and its tangential component u'_T because, as we will see, these variables play different roles in our approach. In the examples below we comment on the friction condition (6). We need the following basic hypothesis.

HYPOTHESIS $H(j)$. *$j : \Gamma_C \times (0,T) \times (\mathbb{R}^d)^3 \to \mathbb{R}$ is a function satisfying*

(i) *$j(\cdot,\cdot,\xi,\zeta,\eta)$ is measurable for all ξ, ζ, $\eta \in \mathbb{R}^d$ and $j(\cdot,\cdot,\xi,\zeta,0) \in L^1(\Gamma_C \times (0,T))$;*
(ii) *$j(x,t,\cdot,\cdot,\eta)$ is continuous for $(x,t) \in \Gamma_C \times (0,T)$, $\eta \in \mathbb{R}^d$;*
(iii) *$j(x,t,\xi,\zeta,\cdot)$ is locally Lipschitz for $(x,t) \in \Gamma_C \times (0,T)$, ξ, $\zeta \in \mathbb{R}^d$;*
(iv) *$\|\partial j(x,t,\xi,\zeta,\eta)\|_{\mathbb{R}^d} \leq c_1(1 + \|\eta\|_{\mathbb{R}^d})$ for all $(x,t) \in \Gamma_C \times (0,T)$, ξ, ζ, $\eta \in \mathbb{R}^d$ with $c_1 > 0$;*
(v) *$j^0(x,t,\xi,\zeta,\eta;-\eta) \leq d_1(1 + \|\eta\|_{\mathbb{R}^d})$ for all (x,t,ξ,ζ,η) with $d_1 \geq 0$,*

where ∂j denotes the Clarke subdifferential of j with respect to the variable η.

EXAMPLE 2. CONTACT WITH NONMONOTONE FRICTION LAWS. Consider first, for simplicity, the case when the function j is independent of (ξ,ζ) and it is nonconvex in η. This is a case of nonmonotone friction laws which are independent of the slip displacement and the slip rate. The friction law (6) takes the form

$$-\sigma_T(t) \in \partial j\left(x,t,u'_T\right), \qquad \text{on } \Gamma_C \times (0,T). \tag{8}$$

This law appears (cf. Section 7.2 of [13]) in the tangential direction of an adhesive interface and describes the partial cracking and crushing of the adhesive bonding material. We refer also to Section 2.4 of [13] for several examples of the zig-zag friction laws which can be formulated in form (8). As a model example we consider a nonconvex function $j : \mathbb{R} \to \mathbb{R}$ given by $j(r) = \min\{j_1(r), j_2(r)\}$, where $j_1(r) = ar^2$, $j_2(r) = (a/2)(r^2+1)$, $a > 0$ (for simplicity we also drop the (x,t)-dependence). Its subdifferential has a form

$$\partial j(r) = \begin{cases} ar, & \text{if } r \in (-\infty,-1) \cup (1,+\infty), \\ 2ar, & \text{if } r \in (-1,1), \\ [a,2a], & \text{if } r = 1, \\ [-2a,-a], & \text{if } r = -1. \end{cases}$$

Using Theorem 2.5.1 of [29] we know that $\partial j(r) \subset \operatorname{co}\{j'_1(r), j'_2(r)\}$ and hence the subdifferential has at most linear growth. From Proposition 2.1.2 of [29] we have $j^0(r;s) = \max\{r^*s : r^* \in \partial j(r)\}$. Hence $j^0(r;-r) = \max\{-r^*r : r^* = \lambda j'_1(r) + (1-\lambda)j'_2(r),\ \lambda \in (0,1)\} \leq 0$ since $j'_k(r)r \geq 0$ for $k = 1,2$. Thus the function j satisfies $H(j)$. Furthermore, if $j : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then $\partial j(r) = j'(r)$ for $r \in \mathbb{R}$ and (8) reduces to the equation $-\sigma_T(t) = j'(u'_T(t))$ on $\Gamma_C \times (0,T)$. When $j(r) = (b/2)r^2$ (b represents the fixed friction coefficient), law (8) takes the form $-\sigma_T(t) = bu'_T(t)$ on $\Gamma_C \times (0,T)$, which simply means that the tangential shear is proportional to the tangential velocity. We remark that the nonconvex superpotential j for $\Omega \subset \mathbb{R}^3$ is formulated by extending to $\mathbb{R}^2$ or to $\mathbb{R}^3$ certain one-dimensional nonmonotone multivalued laws, e.g., by considering minimum type and maximum type functions (cf. Section 4.6.1 of [30] for concrete examples).

Analogously to law (8) we consider the nonmonotone friction conditions which depend on slip and slip rate. In this case we choose $j(x,t,\xi,\zeta,\eta) = g(x,t,\xi,\zeta)h(\eta)$, where $g : \Gamma_C \times (0,T) \times$

$(\mathbb{R}^d)^2 \to \mathbb{R}$ satisfies $g(\cdot,\cdot,\xi,\zeta) \in L^1(\Gamma_C \times (0,T))$, $g(x,t,\cdot,\cdot)$ is continuous, $|g(x,t,\xi,\zeta)| \le g_0$ with a positive constant g_0 and $h : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz, $\|\partial h(\eta)\|_{\mathbb{R}^d} \le c_2(1 + \|\eta\|_{\mathbb{R}^d})$ and $h^0(\eta; -\eta) \le d_2(1 + \|\eta\|_{\mathbb{R}^d})$ with $c_2 > 0$ and $d_2 \ge 0$. Then the function j satisfies $H(j)$ and the friction condition (6) takes the form

$$-\sigma_T(t) \in g(x,t,u(t),u'(t))\, \partial h(u'_T(t)), \qquad \text{on } \Gamma_C \times (0,T). \tag{9}$$

Taking in (9) a suitable function g and the convex function $h(\eta) = \|\eta\|_{\mathbb{R}^d}$, we obtain a number of well-known monotone friction laws which are formulated below.

Example 3. Contact with Simplified Coulomb's Friction Law. Consider a viscoelastic contact problem modeled by a simplified version of Coulomb's law of dry friction, that is,

$$\begin{aligned} &-\sigma_N(t) = S(t), \\ &\quad \|\sigma_T\| \le \mu|\sigma_N|, \qquad \text{with} \\ &\quad \|\sigma_T\| < \mu|\sigma_N| \Longrightarrow u'_T = 0, \\ &\quad \|\sigma_T\| = \mu|\sigma_N| \Longrightarrow \sigma_T = -\lambda u'_T, \qquad \text{with some } \lambda \ge 0. \end{aligned}$$

Here $S \in L^\infty(\Gamma_C \times (0,T))$, $S \ge 0$, is a given normal stress and the coefficient of friction $\mu \in L^\infty(\Gamma_C)$ is such that $\mu \ge 0$ a.e. on Γ_C. This law was already used, e.g., in [1,8,9,12,31]. Choosing

$$j(x,t,\xi,\zeta,\eta) = S(x,t)\mu(x)\|\eta\|_{\mathbb{R}^d}$$

the simplified Coulomb's friction law takes the form of (5) and (6). Since $\partial \|\eta\|_{\mathbb{R}^d}$ equals to $\bar{B}(0,1)$ if $\eta = 0$ and $\eta/\|\eta\|_{\mathbb{R}^d}$ if $\eta \ne 0$, condition (6) is equivalent to

$$\begin{aligned} \|\sigma_T(t)\|_{\mathbb{R}^d} &\le S(x,t)\mu(x), && \text{if } u'_T(t) = 0, \\ -\sigma_T(t) &= S(x,t)\mu(x)\frac{u'_T(t)}{\|u'_T(t)\|_{\mathbb{R}^d}}, && \text{if } u'_T(t) \ne 0. \end{aligned}$$

Example 4. Contact with Slip-Dependent Friction. Consider a viscoelastic contact problem with slip-dependent friction. It is modeled with a condition in which the normal stress on the contact surface is prescribed and the coefficient of friction depends on the slip $\|u_T\|$,

$$\begin{aligned} -\sigma_N(t) &= S(t), \\ \|\sigma_T(t)\|_{\mathbb{R}^d} &< \mu\left(t, \|u_T(t)\|_{\mathbb{R}^d}\right) S(t), && \text{if } u'_T(t) = 0, \\ -\sigma_T(t) &= \mu\left(t, \|u_T(t)\|_{\mathbb{R}^d}\right) S(t)\frac{u'_T(t)}{\|u'_T(t)\|_{\mathbb{R}^d}}, && \text{if } u'_T(t) \ne 0. \end{aligned} \tag{10}$$

The physical model of slip-dependent friction was introduced by Rabinowicz [32] in the geophysical context of earthquakes' modeling. This model of friction was studied by Ionescu and Paumier [26,33], Ionescu and Nguyen [24], Ionescu *et al.* [25], Shillor *et al.* in Chapter 10.1 of [11], and Migórski and Ochal [22]. Let the function j be defined by

$$j(x,t,\xi,\zeta,\eta) = S(x,t)\mu\left(x,t,\|\xi_T\|_{\mathbb{R}^d}\right)\|\eta\|_{\mathbb{R}^d}.$$

We observe that if the normal stress $S \in L^\infty(\Gamma_C \times (0,T))$, $S(x,t) \ge 0$, and the friction coefficient satisfies the following.

Hypothesis $H(\mu)$. *$\mu : \Gamma_C \times (0,T) \times \mathbb{R}_+ \to \mathbb{R}_+$ is such that*

(i) *$\mu(\cdot,\cdot,r)$ is measurable for all $r \in \mathbb{R}$;*
(ii) *$\mu(x,t,\cdot)$ is continuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$;*
(iii) *$0 \le \mu(x,t,r) \le \mu_0$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $r \in \mathbb{R}_+$ with $\mu_0 > 0$,*

then $H(j)$ holds and condition (6) reduces to (10). Relations (10) assert the tangential stress is bounded by the normal stress multiplied by the value of the time-dependent friction coefficient $\mu(t, \|u_T(t)\|_{\mathbb{R}^d})$. If such a limit is not attained, sliding does not occur. Otherwise, the friction stress is opposed to the slip rate u'_T and its absolute value depends on the slip. The function μ depends on $x \in \Gamma_C$ to model the local roughness of the contact surface.

EXAMPLE 5. CONTACT WITH A VERSION OF DRY FRICTION LAW. The classical formulation of contact conditions in the frictional contact with normal damped response is as follows:

$$\begin{aligned} -\sigma_N(t) &= p_N(u'_N), \\ \|\sigma_T\| &\le p_T(u'_N), \qquad \text{with} \\ \|\sigma_T\| &< p_T(u'_N), && \text{if } u'_T = 0, \\ \|\sigma_T\| &= p_T(u'_N)\frac{u'_T(t)}{\|u'_T(t)\|_{\mathbb{R}^d}}, && \text{if } u'_T(t) \neq 0. \end{aligned}$$

There is a number of ways we may choose functions p_N and p_T (see, e.g., Chapter 8.6 of [11]). Let $p_N(x,r) = S(x)$, where $S \in L^\infty(\Gamma_C)$ is a given positive function (cf. (8.6.9) in [11]), i.e., the normal stress is prescribed on Γ_C. This type of contact condition in which the normal stress is given arises in the study of some mechanisms and was considered by, e.g., [1,12]. We choose $j(x,t,\xi,\zeta,\eta) = p_T(x,t,\zeta_N)\|\eta\|_{\mathbb{R}^d}$, where $p_T : \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}_+$ satisfies $p_T(\cdot,\cdot,r) \in L^1(\Gamma_C \times (0,T))$, $p_T(x,t,\cdot)$ is continuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$ and $0 \le p_T(x,t,r) \le p_0$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $r \in \mathbb{R}$ with $p_0 > 0$. Then the friction condition (6) reads

$$-\sigma_T(t) \in p_T\left(x,t,u'_N(t)\right) \partial\|u'_T(t)\|_{\mathbb{R}^d}, \qquad \text{on } \Gamma_C \times (0,T),$$

which is equivalent to

$$\begin{aligned} \|\sigma_T\| &< p_T(t,u'_N), && \text{if } u'_T = 0, \\ \|\sigma_T\| &= p_T(t,u'_N)\frac{u'_T(t)}{\|u'_T(t)\|_{\mathbb{R}^d}}, && \text{if } u'_T(t) \neq 0. \end{aligned}$$

EXAMPLE 6. CONTACT WITH SLIP RATE DEPENDENT FRICTION. Let the function j be given by

$$j(x,t,\xi,\zeta,\eta) = h(x,t,\xi_N)\mu\left(x,t,\|\zeta_T\|_{\mathbb{R}^d}\right)\|\eta\|_{\mathbb{R}^d}.$$

We admit the following assumption.

ASSUMPTION $H(h)$. *$h : \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}_+$ is a function such that*

(i) *$h(\cdot,\cdot,r)$ is measurable for all $r \in \mathbb{R}$;*
(ii) *$h(x,t,\cdot)$ is continuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$;*
(iii) *$0 \le h(x,t,r) \le h_0$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $r \in \mathbb{R}$ with $h_0 > 0$.*

A simple verification shows that if $H(h)$ and $H(\mu)$ (see Example 4) hold, then j satisfies $H(j)$. The condition friction (6) takes the form

$$\begin{aligned} \|\sigma_T(t)\|_{\mathbb{R}^d} &\le h(x,t,u_N)\mu(x,t,0), && \textit{if } u'_T(t) = 0, \\ -\sigma_T(t) &= h(x,t,u_N)\mu\left(x,t,\|u'_T(t)\|_{\mathbb{R}^d}\right)\frac{u'_T(t)}{\|u'_T(t)\|_{\mathbb{R}^d}}, && \textit{if } u'_T(t) \neq 0. \end{aligned}$$

Since μ is a function of u'_T the friction model is slip rate or velocity dependent. In most geological publications dealing with earthquakes the friction coefficient is assumed to depend on the slip rate. For more details on the interpretation of this friction law, we refer, e.g., to [25,26,32] and the references therein.

4. VARIATIONAL FORMULATION

The goal of this section is to give a variational formulation of problem (1)–(7), and state the assumptions and the main result of this paper.

Let V be the closed subspace of $H^1(\Omega;\mathbb{R}^d)$ given by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_D\}.$$

On V we consider the inner product and the corresponding norm defined by

$$\langle u,v\rangle_V = \langle \varepsilon(u),\varepsilon(v)\rangle_{\mathcal{H}}, \quad \|v\| = \|\varepsilon(v)\|_{\mathcal{H}}, \qquad \text{for } u,v \in V.$$

Using the Korn inequality $\|v\|_{H_1} \le c\|\varepsilon(v)\|_{\mathcal{H}}$ for $v \in V$ with $c > 0$ (cf. [1]), we infer that $\|\cdot\|$ and $\|\cdot\|_{H_1}$ are the equivalent norms on V. Moreover, $(V, \|\cdot\|)$ is a Hilbert space. In what follows we identify H with its dual. We have an evolution triple of spaces $V \subset H \subset V^*$ with dense, continuous, and compact embeddings (cf. [27,28]). We denote by $\|\cdot\|_{V^*}$ the norm in V^* and by $\langle\cdot,\cdot\rangle_{V^*\times V}$ the duality of V and V^*. Let $\mathcal{V} = L^2(0,T;V)$, $\hat{\mathcal{H}} = L^2(0,T;H)$, and $\mathcal{W} = \{w \in \mathcal{V} : w' \in \mathcal{V}^*\}$, where $\mathcal{V}^* = L^2(0,T;V^*)$. The space $\mathcal{W}$ equipped with the norm $\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}^*}$ is a separable reflexive Banach space. We also have $\mathcal{W} \subset \mathcal{V} \subset \hat{\mathcal{H}} \subset \mathcal{V}^*$ and the embeddings $\mathcal{W} \subset C(0,T;H)$ and $\{w \in \mathcal{V} : w' \in \mathcal{W}\} \subset C(0,T;V)$ are continuous. Finally, the duality between $\mathcal{V}$ and $\mathcal{V}^*$ is given by $\langle\langle z,w\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^T \langle z(s),w(s)\rangle_{V^*\times V}\,ds$.

Now we obtain the weak formulation of problem (1)–(7). To this end we multiply the equation of motion (1) by $v \in V$ and use the Green formula. We get

$$\langle u''(t),v\rangle_{V^*\times V} + \langle \sigma(t),\varepsilon(v)\rangle_{\mathcal{H}} - \int_\Gamma \sigma(t)n\cdot v\,d\Gamma = \langle f_0(t),v\rangle_H \tag{11}$$

for $v \in V$ and a.e. $t \in (0,T)$. From boundary conditions (3)–(5), we have

$$\int_\Gamma \sigma(t)n\cdot v\,d\Gamma = \int_{\Gamma_N} f_1(t)\cdot v\,d\Gamma + \int_{\Gamma_C} \sigma_T(t)\cdot v_T\,d\Gamma - \int_{\Gamma_C} S(t)v_N\,d\Gamma, \tag{12}$$

and from (6), we deduce

$$-\sigma_T(t)\cdot\eta \le j^0\left(x,t,u(t),u'(t),u'_T(t);\eta\right), \qquad \text{for all } \eta \in \mathbb{R}^d. \tag{13}$$

Using these notations and relations one obtains the following variational formulation of problem (1)–(7): find a displacement field $u : (0,T) \to V$ such that $u \in \mathcal{V}$, $u' \in \mathcal{W}$, and

$$\begin{gathered}\langle u''(t),v\rangle_{V^*\times V} + \langle \mathcal{C}\varepsilon\left(u'(t)\right) + \mathcal{G}\varepsilon(u(t)),\varepsilon(v)\rangle_{\mathcal{H}} + \int_{\Gamma_C} j^0\left(x,t,u(t),u'(t),u'_T;v_T\right)\,d\Gamma(x)\\ \ge \langle f(t),v\rangle_{V^*\times V}, \qquad \text{for all } v \in V, \quad \text{and a.e. } t \in (0,T),\\ u(0) = u_0, \qquad u'(0) = u_1,\end{gathered} \tag{14}$$

where

$$\langle f(t),v\rangle_{V^*\times V} = \langle f_0(t),v\rangle_H + \int_{\Gamma_N} f_1(t)\cdot v\,d\Gamma - \int_{\Gamma_C} S(t)v_N\,d\Gamma.$$

We remark that inequality problem (14) involves the generalized directional derivative of Clarke of a locally Lipschitz function. Such formulation is called a hemivariational inequality (cf. [12,13,34]).

In the study of problem (14) we need the following assumptions.

ASSUMPTION $H(\mathcal{C})$. *The viscosity operator $\mathcal{C} : Q \times \mathcal{S}_d \to \mathcal{S}_d$ satisfies the Carathéodory condition (i.e., $\mathcal{C}(\cdot,\cdot,\varepsilon)$ is measurable on Q for all $\varepsilon \in \mathcal{S}_d$ and $\mathcal{C}(x,t,\cdot)$ is continuous on $\mathcal{S}_d$ for a.e. $(x,t) \in Q$) and*

(i) $\|\mathcal{C}(x,t,\varepsilon)\|_{\mathcal{S}_d} \le c_1(b(x,t) + \|\varepsilon\|_{\mathcal{S}_d})$ *for* $\varepsilon \in \mathcal{S}_d$, a.e. $(x,t) \in Q$ *with* $b \in L^2(Q)$, $c_1 > 0$;
(ii) $(\mathcal{C}(x,t,\varepsilon_1) - \mathcal{C}(x,t,\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \ge 0$ *for all* ε_1, $\varepsilon_2 \in \mathcal{S}_d$ *and* a.e. $(x,t) \in Q$;
(iii) $\mathcal{C}(x,t,\varepsilon) : \varepsilon \ge c_2\|\varepsilon\|^2_{\mathcal{S}_d}$ *for all* $\varepsilon \in \mathcal{S}_d$ *and* a.e. $(x,t) \in Q$ *with* $c_2 > 0$.

ASSUMPTION $H(\mathcal{G})$. *The elasticity operator $\mathcal{G} : \Omega \times \mathcal{S}_d \to \mathcal{S}_d$ is of the form $\mathcal{G}(x,\varepsilon) = \mathbb{E}(x)\varepsilon$ (Hooke's law) with a symmetric and positive elasticity tensor $\mathbb{E} \in L^\infty(\Omega)$, i.e., $\mathbb{E} = (g_{ijkl})$, $i,j,k,l = 1,\dots,d$ with $g_{ijkl} = g_{jikl} = g_{lkij}$ and $g_{ijkl}(x)\chi_{ij}\chi_{kl} \ge 0$ for a.e. $x \in \Omega$ and for all symmetric tensors $\chi = \{\chi_{ij}\}$.*

Furthermore, the forces, the load, and the initial data have the following regularity.

ASSUMPTION $H(f)$. $f_0 \in L^2(0,T;H)$, $f_1 \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$, $S \in L^\infty(\Gamma_C \times (0,T))$, $S \ge 0$, $u_0 \in V$, $u_1 \in H$.

Before we state and prove a result on the existence of solutions to problem (14) we shall examine basic properties of operators and functions which appear in (14). We define the operators $A : (0,T) \times V \to V^*$ and $B : V \to V^*$ by

$$\langle A(t,u), v\rangle_{V^*\times V} = \langle \mathcal{C}(x,t,\varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}}, \qquad \text{for } u,v \in V \quad \text{and} \quad t \in (0,T), \tag{15}$$

$$\langle Bu, v\rangle_{V^*\times V} = \langle \mathcal{G}(x,\varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}}, \qquad \text{for } u,v \in V. \tag{16}$$

LEMMA 7. *If Hypothesis $H(\mathcal{C})$ holds, then the operator $A : (0,T) \times V \to V^*$ defined by (15) satisfies the following.*

HYPOTHESIS $H(A)$. *$A : (0,T) \times V \to V^*$ is such that*

(i) $A(\cdot, v)$ *is measurable on* $(0,T)$ *for all* $v \in V$;
(ii) $A(t,\cdot)$ *is pseudomonotone for every* $t \in (0,T)$;
(iii) $\|A(t,v)\|_{V^*} \le a(t) + b\|v\|$ a.e. t, *for all* $v \in V$ *with* $a \in L^2(0,T)$, $a \ge 0$, $b > 0$;
(iv) $\langle A(t,v), v\rangle_{V^*\times V} \ge \alpha\|v\|^2$ a.e. $t \in (0,T)$, *for all* $v \in V$ *with* $\alpha > 0$.

LEMMA 8. *If hypothesis $H(\mathcal{G})$ holds, then the operator $B : V \to V^*$ defined by (16) satisfies the following.*

HYPOTHESIS $H(B)$. *$B : V \to V^*$ is a bounded, linear, monotone, and symmetric operator, (i.e., $B \in \mathcal{L}(V,V^*)$, $\langle Bv, v\rangle_{V^*\times V} \ge 0$ for all $v \in V$, $\langle Bv, w\rangle_{V^*\times V} = \langle Bw, v\rangle_{V^*\times V}$, for all v, $w \in V$).*

For the proofs of Lemmata 7 and 8, we refer to [16]. We also observe that if $H(f)$ holds, then the following condition is satisfied.

ASSUMPTION (H_0). $f \in \mathcal{V}^*$, $u_0 \in V$, $u_1 \in H$.

In our main existence result we need to strengthen the hypotheses on the superpotential j. We consider the following.

ASSUMPTION $H(j)_1$. *$j : \Gamma_C \times (0,T) \times (\mathbb{R}^d)^3 \to \mathbb{R}$ satisfies $H(j)$ and*

(i) *either* $j(x,t,\xi,\zeta,\cdot)$ *or* $-j(x,t,\xi,\zeta,\cdot)$ *is regular in the sense of Clarke;*
(ii) $j^0(x,t,\cdot,\cdot,\cdot;v)$ *is upper semicontinuous on* $(\mathbb{R}^d)^3$ *for* $(x,t) \in \Gamma_C \times (0,T)$, $v \in \mathbb{R}^d$,

where j^0 is the directional derivative of $j(x,t,\xi,\zeta,\cdot)$ in the direction v.

REMARK 9. Hypothesis $H(j)_1$ is satisfied, for instance, for the following function:

$$j(x,t,\xi,\zeta,\eta) = g(x,t,\xi,\zeta)h(\eta)$$

where $g : \Gamma_C \times (0,T) \times (\mathbb{R}^d)^2 \to \mathbb{R}$ satisfies $g(\cdot,\cdot,\xi,\zeta) \in L^1(\Gamma_C \times (0,T))$, $g(x,t,\cdot,\cdot)$ is continuous, $0 \le g(x,t,\xi,\zeta) \le g_0$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all ξ, $\zeta \in \mathbb{R}^d$ with $g_0 > 0$ and $h : \mathbb{R}^d \to \mathbb{R}$

which is locally Lipschitz and regular, $\|\partial h(\eta)\|_{\mathbb{R}^d} \le c_2(1+\|\eta\|_{\mathbb{R}^d})$ and $h^0(\eta;-\eta) \le d_2(1+\|\eta\|_{\mathbb{R}^d})$ with $c_2 > 0$ and $d_2 \ge 0$. It is clear that j satisfies $H(j)$ and $j(x,t,\xi,\zeta,\cdot)$ is regular. In order to show $H(j)_1$(ii), for $(\xi_n,\zeta_n,\eta_n) \in (\mathbb{R}^d)^3$, $(\xi_n,\zeta_n,\eta_n) \to (\xi,\zeta,\eta)$, and $v \in \mathbb{R}^d$, we have

$$\begin{aligned}
\limsup j^0(x,t,\xi_n,\zeta_n,\eta_n;v) &= \limsup g(x,t,\xi_n,\zeta_n)h^0(\eta_n;v)\\
&= \limsup \left[(g(x,t,\xi_n,\zeta_n)-g(x,t,\xi,\zeta))h^0(\eta_n;v)+g(x,t,\xi,\zeta)h^0(\eta_n;v)\right]\\
&\le c\|v\|_{\mathbb{R}^d} \limsup |g(x,t,\xi_n,\zeta_n)-g(x,t,\xi,\zeta)| + g(x,t,\xi,\zeta)\limsup h^0(\eta_n;v)\\
&\le g(x,t,\xi,\zeta)h^0(\eta;v) = j^0(x,t,\xi,\zeta,\eta;v).
\end{aligned}$$

In particular, Hypothesis $H(j)_1$ holds in Examples 3–6, where we choose $h(\eta) = \|\eta\|_{\mathbb{R}^d}$ and the following functions: in Example 3 we have $g(x,t,\xi,\zeta) = S(x,t)\mu(x)$ with S and μ as in Example 3; in Example 4 we have $g(x,t,\xi,\zeta) = S(x,t)\mu(x,t,\|\xi_T\|_{\mathbb{R}^d})$, where S and μ satisfy hypotheses of Example 4; in Example 5 we have $g(x,t,\xi,\zeta) = p_T(x,t,\zeta_N)$ with p_T satisfying properties stated in Example 5; and in Example 6 we take $g(x,t,\xi,\zeta) = h(x,t,\xi_N)\mu(x,t,\|\zeta_T\|_{\mathbb{R}^d})$, where h and μ satisfy hypotheses of Example 6.

We define the functional $J : (0,T) \times [L^2(\Gamma_C;\mathbb{R}^d)]^3 \to \mathbb{R}$ by

$$J(t,v,w,z) = \int_{\Gamma_C} j(x,t,v(x),w(x),z_T(x))\, d\Gamma(x) \tag{17}$$

for $t \in (0,T)$ and v, w, $z \in L^2(\Gamma_C;\mathbb{R}^d)$.

LEMMA 10. *Under Hypothesis $H(j)$ the functional J defined by (17) satisfies the following.*

ASSUMPTION $H(J)$. *$J : (0,T) \times [L^2(\Gamma_C;\mathbb{R}^d)]^3 \to \mathbb{R}$ is such that*

(i) *$J(\cdot,v,w,z)$ is measurable for all v, w, $z \in L^2(\Gamma_C;\mathbb{R}^d)$ and $J(\cdot,v,w,0) \in L^1(0,T)$;*
(ii) *$J(t,v,w,\cdot)$ is well defined and locally Lipschitz (in fact, Lipschitz on bounded subsets of $L^2(\Gamma_C;\mathbb{R}^d)$) for all v, $w \in L^2(\Gamma_C;\mathbb{R}^d)$;*
(iii) *$\|\partial J(t,v,w,z)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le \tilde{c}(1+\|z\|_{L^2(\Gamma_C;\mathbb{R}^d)})$ for all v, w, $z \in L^2(\Gamma_C;\mathbb{R}^d)$ with $\tilde{c} > 0$, where the subdifferential of J is taken in the variable z;*
(iv) *$J^0(t,v,w,z;-z) \le \tilde{d}(1+\|z\|_{L^2(\Gamma_C;\mathbb{R}^d)})$ for v, w, $z \in L^2(\Gamma_C;\mathbb{R}^d)$ with some $\tilde{d} \ge 0$;*
(v) *for all v, w, z, $\bar{v} \in L^2(\Gamma_C;\mathbb{R}^d)$, we have*

$$J^0(t,v,w,z;\bar{v}) \le \int_{\Gamma_C} j^0(x,t,v(x),w(x),z_T(x);\bar{v}_T(x))\, d\Gamma(x), \tag{18}$$

where $J^0(t,v,w,z;\bar{v})$ denotes the directional derivative of $J(t,v,w,\cdot)$ at a point z in the direction $\bar{v}$.

If $H(j)_1$ holds, then J satisfies the following.

ASSUMPTION $H(J)_1$. *$J : (0,T) \times [L^2(\Gamma_C;\mathbb{R}^d)]^3 \to \mathbb{R}$ is such that $H(J)$ holds:*

(i) *$J(t,v,w,\cdot)$ or $-J(t,v,w,\cdot)$ is regular, respectively, and in (18) we have equality;*
(ii) *$\partial J(t,\cdot,\cdot,\cdot)$ is usc from $[L^2(\Gamma_C;\mathbb{R}^d)]^3$ to $L^2(\Gamma_C;\mathbb{R}^d)$ endowed with the weak topology, for $t \in (0,T)$, where ∂J always denotes the subdifferential of J with respect to the last variable.*

The proof of Lemma 10 will be given in Section 6. Our main existence result is the following.

THEOREM 11. *Under Hypotheses $H(\mathcal{C})$, $H(\mathcal{G})$, $H(j)_1$, and $H(f)$, the hemivariational inequality (14) has a solution.*

In the proof of this theorem we will use the following surjectivity result (see [27]).

PROPOSITION 12. *If Y is a reflexive, strictly convex Banach space, $L : D(L) \subset Y \to Y^*$ is a linear densely defined maximal monotone operator and $T : Y \to 2^{Y^*} \setminus \{\emptyset\}$ is bounded coercive and pseudomonotone with respect to $D(L)$, then $L + T$ is surjective.*

5. PROOF OF THEOREM 11

The proof of Theorem 11 will be carried out in several steps.

5.1. Evolution Inclusion Formulation

First we consider an evolution inclusion associated with the hemivariational inequality (14). Let $Z = H^\delta(\Omega;\mathbb{R}^d)$ with a fixed $\delta \in (1/2,1)$. We denote by $i : V \to Z$ the embedding injection and by $\gamma : Z \to L^2(\Gamma;\mathbb{R}^d)$ the trace operator. We have $\gamma_0 v = \gamma(iv)$ for all $v \in V$. For simplicity we omit the notation of the embedding and we simply write $\gamma_0 v = \gamma v$ for $v \in V$. Therefore we have $V \subset Z \subset H \subset Z^* \subset V^*$ with all embeddings being compact. This also implies that $\mathcal{W} \subset \mathcal{V} \subset \mathcal{Z} \subset \hat{\mathcal{H}} \subset \mathcal{Z}^* \subset \mathcal{V}^*$, where $\mathcal{Z} = L^2(0,T;Z)$ and $\mathcal{Z}^* = L^2(0,T;Z^*)$ denotes its dual. We consider the following evolution inclusion:

$$\begin{gathered}\text{find } u \in \mathcal{V} \text{ with } u' \in \mathcal{W} \text{ such that}\\ u''(t) + A(t,u'(t)) + Bu(t) + \gamma^*\,\partial J(t,\gamma u(t),\gamma u'(t),\gamma u'(t)) \ni f(t) \text{ a.e. } t,\\ u(0) = u_0, \qquad u'(0) = u_1,\end{gathered} \tag{19}$$

where $\gamma^* : L^2(\Gamma;\mathbb{R}^d) \to Z^*$ denotes the adjoint operator of γ. Here and in what follows the subdifferential of ∂J is always taken with respect to the last variable. We say that a function $u \in \mathcal{V}$ solves (19) if and only if $u' \in \mathcal{W}$ and there exists $\eta \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$ such that

$$\begin{gathered}u''(t) + A(t,u'(t)) + Bu(t) + \gamma^*\eta(t) = f(t), \qquad \text{a.e. } t \in (0,T),\\ \eta(t) \in \partial J(t,\gamma u(t),\gamma u'(t)\gamma u'(t)), \qquad \text{a.e. } t \in (0,T),\\ u(0) = u_0, \qquad u'(0) = u_1.\end{gathered}$$

From Lemmata 7, 8, and 10 we know that A, B, and J satisfy $H(A)$, $H(B)$, and $H(J)_1$, respectively. We start with the following equivalence result.

LEMMA 13. *Under hypothesis $H(J)$(i), (ii), and (v), every solution to problem (19) is a solution to the hemivariational inequality (14). If additionally either $j(x,t,\xi,\zeta,\cdot)$ or $-j(x,t,\xi,\zeta,\cdot)$ is regular, then problems (14) and (19) are equivalent.*

PROOF. Let $u \in \mathcal{V}$ be a solution to (19), i.e., $u''(t) + A(t,u'(t)) + Bu(t) + \gamma^*\eta(t) = f(t)$ for a.e. $t \in (0,T)$ and $\eta(t) \in \partial J(t,\gamma u(t),\gamma u'(t),\gamma u'(t))$ for a.e. $t \in (0,T)$. From the definition of Clarke's subdifferential and (18), for $v \in V$, we have

$$\begin{aligned}\langle \gamma^*\eta(t), v\rangle_{Z^*\times Z} = \langle \eta(t), \gamma v\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} &\le J^0\left(t,\gamma u(t),\gamma u'(t),\gamma u'(t);\gamma v\right)\\ &\le \int_{\Gamma_C} j^0(x,t,\gamma u(x,t),\gamma u'(x,t),u_T'(x,t);v_T(x))\,d\Gamma(x).\end{aligned}$$

Using the latter in the equality $\langle u''(t) + A(t,u'(t)) + Bu(t), v\rangle_{V^*\times V} + \langle \gamma^*\eta(t), v\rangle_{Z^*\times Z} = \langle f(t), v\rangle_{V^*\times V}$, we deduce that u is a solution to (14).

The regularity of either $j(x,t,\xi,\zeta,\cdot)$ or $-j(x,t,\xi,\zeta,\cdot)$ implies, by Lemma 10, the regularity of $J(t,v,w,\cdot)$ or $-J(t,v,w,\cdot)$, respectively, which in both cases yields the equality in (18). Therefore, in such cases, every solution to (14) solves also (19). This completes the proof of the proposition. ∎

5.2. Estimate on Solutions

We observe that under the assumptions of Theorem 11, in order to obtain the thesis, it is enough to show an existence result for the evolution inclusion (19). We begin the study of (19) with the *a priori* estimates for its solutions.

LEMMA 14. *Under Hypotheses $H(A)$, $H(B)$, $H(J)$, and (H_0), if u is a solution to (19), then*

$$\|u\|_{C(0,T;V)} + \|u'\|_{\mathcal{W}} \le C\left(1 + \|u_0\| + \|u_1\|_H + \|f\|_{\mathcal{V}^*}\right) \tag{20}$$

with a positive constant C.

PROOF. Let u be a solution to (19). Since $u'(t) \in V$ we take the duality brackets with $u'(t) \in V$ and integrating over $(0,t)$ for any $t \in (0,T)$, we have

$$\int_0^t \langle u''(s), u'(s)\rangle_{V^*\times V}\, ds + \int_0^t \langle A(s,u'(s)), u'(s)\rangle_{V^*\times V}\, ds + \int_0^t \langle Bu(s), u'(s)\rangle_{V^*\times V}\, ds$$
$$+ \int_0^t \langle \xi(s), u'(s)\rangle_{V^*\times V}\, ds = \int_0^t \langle f(s), u'(s)\rangle_{V^*\times V}\, ds$$

with $\xi(s) \in \gamma^*\partial J(s, \gamma u(s), \gamma u'(s), \gamma u'(s))$ for a.e. $s \in (0,t)$. From the integration by parts formula (Proposition 23.23 of [28]), we get $\int_0^t \langle u''(s), u'(s)\rangle_{V^*\times V}\, ds = (1/2)\|u'(t)\|_H^2 - (1/2)\|u_1\|_H^2$. By the monotonicity and symmetry of B, it follows that

$$\begin{aligned}\int_0^t \langle Bu(s), u'(s)\rangle_{V^*\times V}\, ds &= \frac{1}{2}\int_0^t \frac{d}{ds}\langle Bu(s), u(s)\rangle_{V^*\times V}\, ds \\ &= \frac{1}{2}\langle Bu(t), u(t)\rangle_{V^*\times V} - \frac{1}{2}\langle Bu_0, u_0\rangle_{V^*\times V} \ge -\frac{1}{2}\|B\|_{\mathcal{L}(V,V^*)}\|u_0\|^2.\end{aligned}$$

Moreover, from the Young inequality, we get

$$\int_0^t \langle f(s), u'(s)\rangle_{V^*\times V}\, ds \le \int_0^t \|f(s)\|_{V^*}\, \|u'(s)\|\, ds \le \frac{\alpha}{2}\, \|u'\|^2_{L^2(0,t;V)} + \frac{1}{2\alpha}\|f\|^2_{\mathcal{V}^*}$$

for $\alpha > 0$. Keeping in mind the above bounds and exploiting the coercivity of A (see $H(A)$(iv)), we have

$$\begin{aligned}\frac{1}{2}\|u'(t)\|_H^2 - \frac{1}{2}\|u_1\|_H^2 + \alpha\|u'\|^2_{L^2(0,t;V)} - \frac{1}{2}\|B\|\|u_0\|^2 + \int_0^t \langle \xi(s), u'(s)\rangle_{V^*\times V}\, ds \\ \le \frac{\alpha}{2}\|u'\|^2_{L^2(0,t;V)} + \frac{1}{2\alpha}\|f\|^2_{\mathcal{V}^*}\end{aligned} \tag{21}$$

for all $t \in (0,T)$, where $\xi(s) = \gamma^* w(s)$ and $w(s) \in \partial J(s, \gamma u(s), \gamma u'(s), \gamma u'(s))$ for a.e. $s \in (0,t)$. From $H(J)$(iv) we have

$$\begin{aligned}-\langle w(s), \gamma u'(s)\rangle_{L^2(\Gamma;\mathbb{R}^d)} &\le J^0\left(s, \gamma u(s), \gamma u'(s), \gamma u'(s); -\gamma u'(s)\right) \\ &\le \tilde{d}\left(1 + \|\gamma u'(s)\|_{L^2(\Gamma;\mathbb{R}^d)}\right) \le \tilde{d}\left(1 + \beta\|\gamma\|\|u'(s)\|\right)\end{aligned}$$

for a.e. $s \in (0,T)$, where $\beta > 0$ is such that $\|v\|_Z \le \beta\|v\|$ for $v \in V$, $\tilde{d} \ge 0$, and $\|\gamma\| = \|\gamma\|_{\mathcal{L}(Z, L^2(\Gamma;\mathbb{R}^d))}$. Hence, we obtain

$$\begin{aligned}\int_0^t \langle \xi(s), u'(s)\rangle_{V^*\times V}\, ds &= \int_0^t \langle \xi(s), u'(s)\rangle_{Z^*\times Z}\, ds = \int_0^t \langle w(s), \gamma u'(s)\rangle_{L^2(\Gamma;\mathbb{R}^d)}\, ds \\ &\ge -\tilde{d}\int_0^t \left(1 + \beta\|\gamma\|\, \|u'(s)\|\right) ds \ge -\tilde{d}\left(T + \beta\|\gamma\|\sqrt{T}\, \|u'\|_{L^2(0,t;V)}\right).\end{aligned}$$

Combining this inequality with (21), it follows that

$$\frac{1}{2}\|u'(t)\|_H^2 + \frac{\alpha}{2}\|u'\|^2_{L^2(0,t;V)} \le \frac{1}{2}\|u_1\|_H^2 + \frac{1}{2}\|B\|\|u_0\|^2 + \frac{1}{2\alpha}\|f\|^2_{\mathcal{V}} + \tilde{d}T + \tilde{d}\beta\sqrt{T}\|\gamma\|\|u'\|_{L^2(0,t;V)}$$

for all $t \in (0,T)$. Thus, we obtain

$$\|u'\|_{L^2(0,t;V)} \le \tilde{c_2}\,(1 + \|u_0\| + \|u_1\|_H + \|f\|_{\mathcal{V}^*}), \qquad \text{with some } \tilde{c_2} > 0. \tag{22}$$

Hence and from the equality $u(t) = u_0 + \int_0^t u'(s)\,ds$ we get

$$\|u(t)\| \le \|u_0\| + \int_0^t \|u'(s)\|\,ds \le \|u_0\| + \sqrt{T}\tilde{c_2}\,(1 + \|u_0\| + \|u_1\|_H + \|f\|_{\mathcal{V}^*})$$

which implies

$$\|u\|_{C(0,T;V)} \le c_3\,(1 + \|u_0\| + \|u_1\|_H + \|f\|_{\mathcal{V}^*}), \qquad \text{with some } c_3 > 0. \tag{23}$$

To end the proof it is enough to show the estimate on $\|u''\|_{\mathcal{V}^*}$. Since u is a solution to (19), from $H(A)$(iii), $H(B)$, and $H(J)$(iii), we have

$$\|u''\|_{\mathcal{V}^*} \le \|f\|_{\mathcal{V}^*} + \bar{a}_1 + \bar{b}_1\|u'\|_{\mathcal{V}} + \|B\|\|u\|_{\mathcal{V}} + \tilde{\beta}\bar{c}\|\gamma\|\,(1 + \beta\|\gamma\|\|u'\|_{\mathcal{V}}), \tag{24}$$

where $\tilde{\beta} > 0$ is the embedding constant of $\mathcal{Z}^*$ into $\mathcal{V}^*$, $\bar{a}_1 = \sqrt{2}\|a_1\|_{L^2(0,T)}$, and $\bar{b}_1 = \sqrt{2}b_1^2$. Combining (22)–(24), we obtain estimate (20), which completes the proof of the proposition. ∎

5.3. Abstract Operator Inclusion for $u_1 \in V$

Next we shall reformulate the evolution inclusion (19) as an abstract operator inclusion. To this end let us define the operator $K : \mathcal{V} \to C(0,T;V)$ by $Kv(t) = \int_0^t v(s)\,ds + u_0$ for $v \in \mathcal{V}$. Problem (19) can be formulated as follows:

$$\begin{gathered}\text{find } z \in \mathcal{W} \text{ such that}\\ z'(t) + A(t,z(t)) + B(Kz(t)) + \gamma^*\partial J(t,\gamma Kz(t),\gamma z(t),\gamma z(t)) \ni f(t), \qquad \text{a.e } .t \in (0,T),\\ z(0) = u_1,\end{gathered} \tag{25}$$

and we remark that z is a solution to (25) if and only if $u = Kz$ satisfies (19). In what follows we consider two cases: first suppose that $u_1 \in V$ and then we pass to the more general case $u_1 \in H$.

Let us assume that $u_1 \in V$ and define the following operators: $\mathcal{A}_1 : \mathcal{V} \to \mathcal{V}^*$, $\mathcal{B}_1 : \mathcal{V} \to \mathcal{V}^*$, and $\mathcal{N}_1 : \mathcal{V} \to 2^{\mathcal{V}^*}$ by

$$(\mathcal{A}_1 v)(t) = A(t, v(t) + u_1), \tag{26}$$

$$(\mathcal{B}_1 v)(t) = BK(v(t) + u_1), \tag{27}$$

$$\begin{aligned}\mathcal{N}_1 v = \{w \in \mathcal{Z}^* : w(t) \in \gamma^*\partial J(t, K(v(t)+u_1), \gamma(v(t)+u_1), \gamma(v(t)+u_1)),\\ \text{a.e } .t \in (0,T)\},\end{aligned} \tag{28}$$

for $v \in \mathcal{V}$. We observe that $\mathcal{A}_1 v = \mathcal{A}(v + u_1)$ and $\mathcal{B}_1 v = \mathcal{B}(K(v + u_1))$, where $\mathcal{A}$ and $\mathcal{B}$ are the Nemitsky operators corresponding to A and B, respectively, i.e.,

$$(\mathcal{A}v)(t) = A(t, v(t)), \quad (\mathcal{B}v)(t) = B(v(t)), \qquad \text{for } v \in \mathcal{V}. \tag{29}$$

From (25) we immediately get

$$\begin{gathered} z' + \mathcal{A}_1 z + \mathcal{B}_1 z + \mathcal{N}_1 z \ni f,\\ z(0) = 0,\end{gathered} \tag{30}$$

and observe that $z \in \mathcal{W}$ solves (25) if and only if $z - u_1 \in \mathcal{W}$ is a solution to (30).

Let the operator $L : D(L) \subset \mathcal{V} \to \mathcal{V}^*$ be defined by $Lv = v'$ with $D(L) = \{v \in \mathcal{W} : v(0) = 0\}$. Recall (cf. [28, Proposition 32.10]) that L is a linear, densely defined, and maximal monotone operator. Now problem (30) can be written as

$$\text{find } z \in D(L) \text{ such that } (L + \mathcal{T})z \ni f,$$

where $\mathcal{T} : \mathcal{V} \to 2^{\mathcal{V}^*}$ is given by $\mathcal{T}v = (\mathcal{A}_1 + \mathcal{B}_1 + \mathcal{N}_1)v$ for $v \in \mathcal{V}$.

In order to establish the existence of solutions to (30) we will show that the operator $\mathcal{T}$ is bounded, coercive, and pseudomonotone with respect to $D(L)$, and apply Proposition 12. The following three auxiliary results show the properties of the operators $\mathcal{A}_1$, $\mathcal{B}_1$, and $\mathcal{N}_1$, respectively. The proofs of Lemmata 15 and 16 are given in [16] while the proof of Lemma 17 is postponed to the next section.

LEMMA 15. *If $H(A)$ holds and $u_1 \in V$, then the operator $\mathcal{A}_1$ defined by (26) satisfies:*

(a) $\|\mathcal{A}_1 v\|_{\mathcal{V}^*} \le \hat{a}_1 + \hat{b}_1 \|v\|_{\mathcal{V}}$ *for all* $v \in \mathcal{V}$ *with* $\hat{a}_1 \ge 0$ *and* $\hat{b}_1 > 0$;

(b) $\langle\langle \mathcal{A}_1 v, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \ge (\alpha/2)\|v\|^2_{\mathcal{V}} - \hat{\beta}_2 \|v\|_{\mathcal{V}} - \hat{\beta}_3$ *for all* $v \in \mathcal{V}$ *with* $\hat{\beta}_2 \ge 0$ *and* $\hat{\beta}_3 \ge 0$;

(c) $\mathcal{A}_1$ *is demicontinuous, i.e., for any sequence* $\{v_n\} \subset \mathcal{V}$, $v_n \to v$ *in* $\mathcal{V}$, *we have* $\mathcal{A}_1 v_n \to \mathcal{A}_1 v$ *weakly in* $\mathcal{V}^*$;

(d) $\mathcal{A}_1$ *is L-pseudomonotone.*

If $H(A)$ holds, then the operator $\mathcal{A}$ defined by (29) satisfies:

(e) *for every sequence* $\{v_n\} \subset \mathcal{W}$ *with* $v_n \to v$ *weakly in* $\mathcal{W}$ *and* $\limsup \langle\langle \mathcal{A} v_n, v_n - v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \le 0$, *it follows that* $\mathcal{A} v_n \to \mathcal{A} v$ *weakly in* $\mathcal{V}^*$ *and* $\langle\langle \mathcal{A} v_n, v_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle\langle \mathcal{A} v, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}}$.

LEMMA 16. *If $H(B)$ holds and $u_1 \in V$, then the operator $\mathcal{B}_1$ defined by (27) satisfies:*

(a) $\|\mathcal{B}_1 v\|_{\mathcal{V}^*} \le \hat{c}_1 (1 + \|v\|_{\mathcal{V}})$ *for all* $v \in \mathcal{V}$ *with* $\hat{c}_1 > 0$;

(b) $\|\mathcal{B}_1 v - \mathcal{B}_1 w\|_{\mathcal{V}^*} \le \hat{c}_2 \|v - w\|_{\mathcal{V}}$ *for all* $v, w \in \mathcal{V}$ *with* $\hat{c}_2 > 0$;

(c) $\langle\langle \mathcal{B}_1 v, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \ge -\hat{c}_3 \|v\|_{\mathcal{V}} - \hat{c}_4$ *for all* $v \in \mathcal{V}$ *with* $\hat{c}_3 \ge 0$ *and* $\hat{c}_4 \ge 0$;

(d) $\mathcal{B}_1$ *is monotone;*

(e) $\mathcal{B}_1$ *is weakly continuous, i.e., for any sequence* $\{v_n\} \subset \mathcal{V}$ *with* $v_n \to v$ *weakly in* $\mathcal{V}$, *we have* $\mathcal{B}_1 v_n \to \mathcal{B}_1 v$ *weakly in* $\mathcal{V}^*$.

If $H(B)$ holds, then the operator $\mathcal{B}$ defined by (29) satisfies:

(f) $\langle\langle \mathcal{B} v, v' \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \ge 0$ *for all* $v \in \mathcal{V}$ *such that* $v' \in \mathcal{W}$ *and* $v(0) = 0$.

LEMMA 17. *If $H(J)$ holds and $u_1 \in V$, then the operator $\mathcal{N}_1$ defined by (28) satisfies:*

(a) $\|z\|_{\mathcal{Z}^*} \le \bar{c}(1 + \|v\|_{\mathcal{V}})$ *for all* $z \in \mathcal{N}_1 v$ *and* $v \in \mathcal{V}$ *with* $\bar{c} > 0$;

(b) *for every* $v \in \mathcal{V}$, $\mathcal{N}_1 v$ *is a nonempty convex and weakly compact subset of* $\mathcal{Z}^*$;

(c) $\langle\langle z, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \ge -\tilde{c}_1 \|v\|_{\mathcal{V}} - \tilde{c}_2$ *for all* $z \in \mathcal{N}_1 v$ *and* $v \in \mathcal{V}$ *with* $\tilde{c}_1, \tilde{c}_2 > 0$.

If $H(J)_1$ holds and $u_1 \in V$, then additionally we have:

(d) *for every* v_n, $v \in \mathcal{V}$ *with* $v_n \to v$ *in* $\mathcal{Z}$ *and every* z_n, $z \in \mathcal{Z}^*$ *with* $z_n \to z$ *weakly in* $\mathcal{Z}^*$, *if* $z_n \in \mathcal{N}_1 v_n$, *then* $z \in \mathcal{N}_1 v$.

We now continue the proof of Theorem 11.

CLAIM 1. *$\mathcal{T}$ is a bounded operator. The fact that the operator $\mathcal{T}$ maps bounded subsets of $\mathcal{V}$ into bounded subsets of $\mathcal{V}^*$ follows from Lemmata 15(a), 16(a), 17(a), and the continuity of the embedding $\mathcal{Z}^* \subset \mathcal{V}^*$.*

CLAIM 2. *$\mathcal{T}$ is coercive. Let $v \in \mathcal{V}$ and $\eta \in \mathcal{T} v$, i.e., $\eta = \mathcal{A}_1 v + \mathcal{B}_1 v + \xi$ with $\xi \in \mathcal{N}_1 v$. From Lemmata 15(b), 16(c), and 17(c), we have*

$$\begin{aligned} \langle\langle \eta, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} &= \langle\langle \mathcal{A}_1 v, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle\langle \mathcal{B}_1 v, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle\langle \xi, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\ge \frac{\alpha}{2} \|v\|^2_{\mathcal{V}} - \hat{\beta}_2 \|v\|_{\mathcal{V}} - \hat{\beta}_3 - \hat{c}_3 \|v\|_{\mathcal{V}} - \hat{c}_4 - \tilde{c} \|v\|_{\mathcal{V}} - \tilde{c}_2. \end{aligned}$$

This yields the coercivity of $\mathcal{T}$.

CLAIM 3. *$\mathcal{T}$ is pseudomonotone with respect to $D(L)$. From Lemma 17(b) it follows that for every $v \in \mathcal{V}$, $\mathcal{T} v$ is a nonempty convex and weakly compact subset of $\mathcal{V}^*$. We show that $\mathcal{T}$ is upper semicontinuous in $\mathcal{V} \times \mathcal{V}^*$, where $\mathcal{V}^*$ is equipped with its weak topology. To this end, we show that if a set K is weakly closed in $\mathcal{V}^*$, then the set*

$$\mathcal{T}^-(K) = \{v \in \mathcal{V} : \mathcal{T} v \cap K \ne \emptyset\} \text{ is closed in } \mathcal{V}.$$

Let $\{v_n\} \subset \mathcal{T}^-(K)$ and assume that $v_n \to v$ in $\mathcal{V}$. We can find $\eta_n \in \mathcal{T} v_n \cap K$ for all $n \in \mathbb{N}$ and by definition we have

$$\eta_n = \mathcal{A}_1 v_n + \mathcal{B}_1 v_n + \xi_n, \qquad \text{with } \xi_n \in \mathcal{N}_1 v_n. \tag{31}$$

Since $\{v_n\}$ is bounded in $\mathcal{V}$ and $\mathcal{T}$ is a bounded operator (by Claim 1), we know that the sequence $\{\eta_n\}$ is bounded in $\mathcal{V}^*$. Hence we may assume that

$$\eta_n \to \eta \text{ weakly in } \mathcal{V}^* \text{ with } \eta \in K, \tag{32}$$

since K is weakly closed in $\mathcal{V}^*$. Moreover, by the boundedness of $\mathcal{N}_1$ (cf. Lemma 17(a)), we know that $\{\xi_n\}$ is bounded in $\mathcal{Z}^*$ and again we may suppose that

$$\xi_n \to \xi \text{ weakly in } \mathcal{Z}^* \text{ with } \xi \in \mathcal{Z}^*. \tag{33}$$

Hence and from the fact that $v_n \to v$ in $\mathcal{Z}$ (recall that $\mathcal{V} \subset \mathcal{Z}$ continuously), by Lemma 17(d), we obtain $\xi \in \mathcal{N}_1 v$. Next, from the demicontinuity of $\mathcal{A}_1$ (cf. Lemma 15(c)) and the continuity of $\mathcal{B}_1$ (cf. Lemma 16(b)), we have

$$\mathcal{A}_1 v_n \to \mathcal{A}_1 v \text{ weakly in } \mathcal{V}^* \quad \text{and} \quad \mathcal{B}_1 v_n \to \mathcal{B}_1 v \text{ in } \mathcal{V}^*.$$

From these convergences, (32) and (33), passing to the limit in (31) we get $\eta = \mathcal{A}_1 v + \mathcal{B}_1 v + \xi$ with $\xi \in \mathcal{N}_1 v$ which means that $\eta \in \mathcal{T}v \cap K$, so $v \in \mathcal{T}^-(K)$. This proves that $\mathcal{T}^-(K)$ is closed in $\mathcal{V}$, and hence $\mathcal{T}$ is upper semicontinuous from $\mathcal{V}$ into $\mathcal{V}^*$ endowed with the weak topology.

To conclude the proof that $\mathcal{T}$ is pseudomonotone with respect to $D(L)$, it is enough to show Condition (d) in the definition of pseudomonotonicity (see Preliminaries). Let $\{z_n\} \subset D(L)$, $z_n \to z$ weakly in $\mathcal{W}$, $\eta_n \in \mathcal{T}z_n$, $\eta_n \to \eta$ weakly in $\mathcal{V}^*$ and assume that

$$\limsup \langle\langle \eta_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \le 0. \tag{34}$$

So $\eta_n = \mathcal{A}_1 z_n + \mathcal{B}_1 z_n + \xi_n$ where $\xi_n \in \mathcal{N}_1 z_n$ for all $n \in \mathbb{N}$.

Since $\mathcal{N}_1$ is a bounded map (cf. Lemma 17(a)) and $\{z_n\}$ is bounded in $\mathcal{V}$, we know that $\{\xi_n\}$ remains in a bounded subset of $\mathcal{Z}^*$. By passing to a subsequence, if necessary, we may suppose that

$$\xi_n \to \xi \text{ weakly in } \mathcal{Z}^*. \tag{35}$$

Since $V \subset Z$ compactly, from the Aubin theorem (see Theorem 3.4.13 in [27]), we have that $\mathcal{W} \subset \mathcal{Z}$ compactly. Thus we may assume that

$$z_n \to z, \qquad \text{in } \mathcal{Z}. \tag{36}$$

From (35), (36), and Lemma 17(d) we deduce that $\xi \in \mathcal{N}_1 z$. From Lemma 17(a) and (36), we have

$$|\langle\langle \xi_n, z_n - z \rangle\rangle_{\mathcal{Z}^* \times \mathcal{Z}}| \le \|\xi_n\|_{\mathcal{Z}^*} \|z_n - z\|_{\mathcal{Z}} \le \bar{c}_5 \left(1 + \|z_n\|_{\mathcal{V}}\right) \|z_n - z\|_{\mathcal{Z}} \to 0 \tag{37}$$

with some $\bar{c}_5 > 0$. On the other hand, by the monotonicity of $\mathcal{B}_1$ (cf. Lemma 16(d)) and (36), we obtain

$$\limsup \langle\langle \mathcal{B}_1 z_n, z - z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \le \limsup \langle\langle \mathcal{B}_1 z, z - z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} = 0. \tag{38}$$

Combining condition (34) with (37) and (38), we have

$$\begin{aligned} \limsup \langle\langle \mathcal{A}_1 z_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} &\le \limsup \langle\langle \eta_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\quad + \limsup \langle\langle \mathcal{B}_1 z_n, z - z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} + \limsup \langle\langle \xi_n, z - z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\le 0. \end{aligned}$$

From the L-pseudomonotonicity of $\mathcal{A}_1$ (cf. Lemma 15(d)), we obtain

$$\mathcal{A}_1 z_n \to \mathcal{A}_1 z \text{ weakly in } \mathcal{V}^* \tag{39}$$

and $\langle\langle \mathcal{A}_1 z_n, z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle\langle \mathcal{A}_1 z, z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}}$ or equivalently

$$\langle\langle \mathcal{A}_1 z_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \to 0. \tag{40}$$

From (39), the weak continuity of $\mathcal{B}_1$ (cf. Lemma 16(e)) and (35), we conclude that

$$\eta_n = \mathcal{A}_1 z_n + \mathcal{B}_1 z_n + \xi_n \to \mathcal{A}_1 z + \mathcal{B}_1 z + \xi = \eta \text{ weakly in } \mathcal{V}^*.$$

The latter together with $\xi \in \mathcal{N}_1 z$ implies $\eta \in \mathcal{T} z$. Moreover, we also have

$$\langle\langle \mathcal{B}_1 z_n, z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle\langle \mathcal{B}_1 z, z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}}. \tag{41}$$

Indeed, from (34), (37), and (40), we get

$$\begin{aligned} \limsup \langle\langle \mathcal{B}_1 z_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} &\le \limsup \langle\langle \eta_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\quad - \lim \langle\langle \mathcal{A}_1 z_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} - \lim \langle\langle \xi_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\le 0 \end{aligned}$$

which together with (38) yields $\lim \langle\langle \mathcal{B}_1 z_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} = 0$, and implies (41). Passing to the limit in the equation

$$\langle\langle \eta_n, z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} = \langle\langle \mathcal{A}_1 z_n, z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle\langle \mathcal{B}_1 z_n, z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle\langle \xi_n, z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}}$$

from (40), (41), and (37), we get $\lim \langle\langle \eta_n, z_n \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle\langle \eta, z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}}$ with $\eta \in \mathcal{T} z$. This proves the pseudomonotonicity of $\mathcal{T}$ with respect to $D(L)$.

Since $\mathcal{V}$ is a strictly convex Banach space, from Claims 1–3, by Proposition 12, we deduce that problem (30) has a solution $z \in D(L)$, so $z + u_1$ solves (25), and $u = K(z + u_1)$ is a solution of (19) in the case when $u_1 \in V$.

5.4. Existence in General Case $u_1 \in H$

In the final step of the proof of Theorem 11, let us suppose now that $u_1 \in H$. Since $V \subset H$ densely, we can find a sequence $\{u_{1n}\} \subset V$ such that $u_{1n} \to u_1$ in H, as $n \to \infty$. Consider a solution u_n of problem (19), when u_1 is replaced by u_{1n}, i.e., a solution of the following problem:

$$\begin{gathered} \text{find } u_n \in \mathcal{V} \text{ such that } u_n' \in \mathcal{W} \quad \text{and} \\ u_n''(t) + A\left(t, u_n'(t)\right) + Bu_n(t) + \gamma^* \left(\partial J\left(t, \gamma u_n(t), \gamma u_n'(t), \gamma u_n'(t)\right)\right) \ni f(t), \qquad \text{a.e. } t, \\ u_n(0) = u_0, \qquad u_n'(0) = u_{1n}. \end{gathered}$$

The existence of u_n, for $n \in \mathbb{N}$, follows from the first part of the proof. Clearly, we have

$$u_n''(t) + A(t, u_n'(t)) + Bu_n(t) + \xi_n(t) = f(t), \qquad \text{for a.e. } t \in (0,T), \tag{42}$$

with

$$\xi_n(t) \in \gamma^* \partial J(t, \gamma u_n(t), \gamma u_n'(t), \gamma u_n'(t)), \qquad \text{for a.e. } t \in (0,T), \tag{43}$$

and the initial conditions $u_n(0) = u_0$, $u_n'(0) = u_{1n}$. Since $\{u_{1n}\}$ is bounded in H, by Lemma 14 we get that the sequences $\{u_n\}$, $\{u_n'\}$ are bounded, respectively, in $\mathcal{V}$ and $\mathcal{W}$ uniformly with respect to n. Hence, by passing to a subsequence if necessary, we assume $u_n \to u$, $u_n' \to u'$ both weakly in $\mathcal{V}$, and $u_n'' \to u''$ weakly in $\mathcal{V}^*$.

We will show that u is a solution to problem (19). Since $u_n \to u$, $u_n' \to u'$ both weakly in $\mathcal{W}$ and $\mathcal{W} \subset C(0,T;H)$ continuously, we get $u_n(t) \to u(t)$ and $u_n'(t) \to u'(t)$ both weakly in H for all $t \in [0,T]$. Hence $u_0 = u_n(0) \to u(0)$ weakly in H, which gives $u(0) = u_0$. Also from the convergences $u_{1n} \to u_1$ in H and $u_{1n} = u_n'(0) \to u'(0)$ weakly in H, we get $u'(0) = u_1$.

Next, from (43) we have $\xi_n(t) = \gamma^* z_n(t)$ and

$$z_n(t) \in \partial J(t, \gamma u_n(t), \gamma u_n'(t), \gamma u_n'(t)), \qquad \text{a.e. } t \in (0,T). \tag{44}$$

Analogously as in Lemma 17(a), by $H(J)$(iii), we know that $\{z_n\}$ remains in a bounded subset of $L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$, and so for a subsequence we may assume

$$z_n \to z \text{ weakly in } L^2\left(0,T;L^2\left(\Gamma_C;\mathbb{R}^d\right)\right). \tag{45}$$

Moreover, we get

$$\xi_n \to \xi \text{ weakly in } \mathcal{Z}^*. \tag{46}$$

Using the last two convergences from $\xi_n(t) = \gamma^* z_n(t)$ for a.e. $t \in (0,T)$, passing to the limit, we have $\xi(t) = \gamma^* z(t)$ a.e. $t \in (0,T)$. Since $\mathcal{W} \subset \mathcal{Z}$ compactly and $u_n \to u$, $u_n' \to u'$ both weakly in $\mathcal{W}$, we know that $u_n \to u$, $u_n' \to u'$ both in $\mathcal{Z}$. For a next subsequence we may suppose $u_n'(t) \to u'(t)$ in Z for a.e. $t \in (0,T)$ and

$$\gamma u_n(t) \to \gamma u(t), \quad \gamma u_n'(t) \to \gamma u'(t), \qquad \text{in } L^2\left(\Gamma_C;\mathbb{R}^d\right), \quad \text{for a.e. } t \in (0,T). \tag{47}$$

Exploiting (45), (47), and the fact that the mapping $\partial J(t,\cdot,\cdot,\cdot)$ is usc from $[L^2(\Gamma_C;\mathbb{R}^d)]^3$ to $L^2(\Gamma_C;\mathbb{R}^d)$ endowed with the weak topology (see Lemma 10), by the convergence theorem of Aubin and Cellina [35], from (44) we have $z(t) \in \partial J(t, \gamma u(t), \gamma u'(t), \gamma u'(t))$ for a.e. $t \in (0,T)$. This implies that

$$\xi(t) \in \gamma^* \partial J(t, \gamma u(t), \gamma u'(t), \gamma u'(t)), \qquad \text{for a.e. } t \in (0,T). \tag{48}$$

Next we will show

$$\mathcal{A}u_n' \to \mathcal{A}u', \text{ weakly in } \mathcal{V}^*. \tag{49}$$

Recall that we use the notation $\mathcal{A}$ and $\mathcal{B}$ for the Nemitsky operators corresponding to A and B, respectively (cf. (29)). Since $\lim\langle\langle \xi_n, u_n' - u'\rangle\rangle_{\mathcal{Z}^*\times\mathcal{Z}} = 0$ ($\xi_n \to \xi$ weakly in $\mathcal{Z}^*$ and $u_n' \to u'$ in $\mathcal{Z}$) and $\lim\langle\langle f, u_n' - u'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0$, from (42), we have

$$\begin{aligned} &\limsup\langle\langle \mathcal{A}u_n', u_n' - u'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &\le \limsup\langle\langle u_n'', u' - u_n'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} + \limsup\langle\langle \mathcal{B}u_n, u' - u_n'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}}. \end{aligned} \tag{50}$$

Using the equality (cf. Proposition 23.23 of [28])

$$\langle\langle u_n'' - u'', u_n' - u'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = \frac{1}{2}\left\|u_n'(T) - u'(T)\right\|_H^2 - \frac{1}{2}\left\|u_n'(0) - u'(0)\right\|_H^2,$$

we get

$$\limsup\langle\langle u_n'' - u'', u' - u_n'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0. \tag{51}$$

By Property (f) of Lemma 16, we have

$$\begin{aligned} \limsup\langle\langle \mathcal{B}u_n, u' - u_n'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \limsup\left(-\langle\langle \mathcal{B}u - \mathcal{B}u_n, u' - u_n'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}}\right. \\ &\qquad \left.+ \langle\langle \mathcal{B}u, u' - u_n'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}}\right) \\ &\le \limsup\langle\langle \mathcal{B}u, u' - u_n'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0. \end{aligned} \tag{52}$$

Hence, by using (51) and (52) in (50), we get $\limsup\langle\langle \mathcal{A}u_n', u_n' - u'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0$. Since $u_n' \to u'$ weakly in $\mathcal{W}$, applying Lemma 15(e) we have (49).

Finally, convergences (46), (49), and the weak continuity of the operator $\mathcal{B}$ (by an argument analogous to that used in Lemma 16(e), cf. [16]), allow us to pass to the limit in the equation $u_n'' + \mathcal{A}u_n' + \mathcal{B}u_n + \xi_n = f$ in $\mathcal{V}^*$ and we obtain $u'' + \mathcal{A}u' + \mathcal{B}u + \xi = f$ in $\mathcal{V}^*$. This together with (48) and initial conditions $u(0) = u_0$ and $u'(0) = u_1$ imply that u is a solution to problem (19). The proof of the theorem is finished. ∎

EXAMPLE 18. The regularity hypothesis in $H(j)_1$ is satisfied, for instance, for a function which is represented as the difference of convex functions. More precisely, let us consider the following condition (for simplicity we omit the (x,t,ξ,ζ)-dependence).

HYPOTHESIS $H(d.c.)$. *The function $j : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz and of d.c.-type, i.e., $j(\xi) = j_1(\xi) - j_2(\xi)$ for $\xi \in \mathbb{R}^d$, where $j_k : \mathbb{R}^d \to \mathbb{R}$, $k = 1, 2$ are convex functions and one of the convex subdifferentials ∂j_k is assumed to be a singleton for every $\xi \in \mathbb{R}^d$ and the growth conditions hold $\|\eta\|_{\mathbb{R}^d} \le c_0(1 + \|\xi\|_{\mathbb{R}^d})$ for $\eta \in \partial j_k(\xi)$ for all $\xi \in \mathbb{R}^d$, $k = 1, 2$, with $c_0 > 0$.*

Under Hypothesis $H(d.c.)$ either j or $-j$ is regular in the sense of Clarke and $\partial j(\xi) = \partial j_1(\xi) - \partial j_2(\xi)$ with $\|\eta\|_{\mathbb{R}^d} \le c_0(1+\|\xi\|_{\mathbb{R}^d})$ for $\eta \in \partial j(\xi)$ for $\xi \in \mathbb{R}^d$, $c_0 > 0$. This can be shown analogously as in [36].

6. PROOFS OF LEMMATA

In this section, we give proofs of Lemmata 10 and 17 on properties of the functional J given by (17) and of the operator $\mathcal{N}_1$ defined by (28).

PROOF OF LEMMA 10. Let v, $w \in L^2(\Gamma_C; \mathbb{R}^d)$ and let $j_1 : \Gamma_C \times (0, T) \times \mathbb{R}^d \to \mathbb{R}$ be defined by $j_1(x, t, \xi) = j(x, t, v(x), w(x), N\xi)$, where $N \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ is given by $N\xi = \xi_T$. Then, from $H(j)$, we get

$$j_1(\cdot, \cdot, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^d,\ j_1(x, t, 0) \in L^1(\Gamma_C \times (0, T)),$$
$$j_1(x, t, \cdot) \text{ is locally Lipschitz for all } (x, t) \in \Gamma_C \times (0, T).$$

From (ii) of Proposition 1 and the fact that $N^* \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ satisfies $N^* = N$, we have

$$\partial j_1(x, t, \xi) = \partial_\xi\left(j(x, t, v(x), w(x), N\xi)\right) \subset N^* \partial_\xi j(x, t, v(x), w(x), N\xi).$$

Hence, by $H(j)$ we get

$$\|\partial j_1(x, t, \xi)\|_{\mathbb{R}^d} \le \tilde{c}\,(1 + \|\xi\|_{\mathbb{R}^d}), \qquad \text{for all } \xi \in \mathbb{R}^d, \quad (x, t) \in \Gamma_C \times (0, T), \quad \text{with } \tilde{c} > 0.$$

Applying Theorem 2.7.5 of Clarke [29], we easily deduce $H(J)$(i),(ii) and

$$\partial J(t, v, w, z) \subset \int_{\Gamma_C} \partial\left(j_1(x, t, z(x))\right) d\Gamma(x) \subset \int_{\Gamma_C} [\partial j(x, t, v(x), w(x), z_T(x))]_T \, d\Gamma(x)$$

for all v, w, $z \in L^2(\Gamma_C; \mathbb{R}^d)$. Next, from the Fatou lemma, we have

$$\begin{aligned}
J^0(t, v, w, z; \bar{v}) &= \limsup_{y \to z, \lambda \downarrow 0} \frac{J(t, v, w, y + \lambda \bar{v}) - J(t, v, w, y)}{\lambda} \\
&= \limsup_{y \to z, \lambda \downarrow 0} \int_{\Gamma_C} \frac{j(x, t, v(x), w(x), y_T + \lambda \bar{v}_T) - j(x, t, v(x), w(x), y_T)}{\lambda} \, d\Gamma(x) \\
&\le \int_{\Gamma_C} \limsup_{y_T \to z_T, \lambda \downarrow 0} \frac{j(x, t, v(x), w(x), y_T + \lambda \bar{v}_T) - j(x, t, v(x), w(x), y_T)}{\lambda} \, d\Gamma(x) \\
&= \int_{\Gamma_C} j_\eta^0(x, t, v(x), w(x), z_T(x); \bar{v}_T(x)) \, d\Gamma(x),
\end{aligned}$$

for all v, w, z, $\bar{v} \in L^2(\Gamma_C; \mathbb{R}^d)$ which gives (18). Using (18) and Hypothesis $H(j)$(v) we obtain inequality $H(J)$(iv).

We assume now that $j(x, t, \xi, \zeta, \cdot)$ is regular, that is, the one-sided directional derivative $j_\eta'(x, t, \xi, \zeta, \eta; \bar{\eta})$ exists and $j_\eta^0(x, t, \xi, \zeta, \eta; \bar{\eta}) = j_\eta'(x, t, \xi, \zeta, \eta; \bar{\eta})$ for all ξ, ζ, η, $\bar{\eta} \in \mathbb{R}^d$. Exploiting again Fatou's lemma and the regularity of j we get

$$\begin{aligned}
J^0(t, v, w, z; \bar{v}) &\ge \liminf_{\lambda \downarrow 0} \frac{J(t, v, w, z + \lambda \bar{v}) - J(t, v, w, z)}{\lambda} \\
&= \liminf_{\lambda \downarrow 0} \int_{\Gamma_C} \frac{j(x, t, v(x), w(x), z_T(x) + \lambda \bar{v}_T(x)) - j(x, t, v(x), w(x), z_T(x))}{\lambda} \, d\Gamma(x) \\
&\ge \int_{\Gamma_C} \liminf_{\lambda \downarrow 0} \frac{j(x, t, v(x), w(x), z_T(x) + \lambda \bar{v}_T(x)) - j(x, t, v(x), w(x), z_T(x))}{\lambda} \, d\Gamma(x) \\
&= \int_{\Gamma_C} j_\eta'(x, t, v(x), w(x), z_T(x); \bar{v}_T(x)) \, d\Gamma(x) \\
&= \int_{\Gamma_C} j_\eta^0(x, t, v(x), w(x), z_T(x); \bar{v}_T(x)) \, d\Gamma(x) \ge J^0(t, v, w, z; \bar{v}).
\end{aligned}$$

Hence $J_z'(t,v,w,z;\bar v)$ exists and $J^0(t,v,w,z;\bar v) = J_z'(t,v,w,z;\bar v)$ for all $\bar v \in L^2(\Gamma_C;\mathbb{R}^d)$. This means that $J(t,v,w,\cdot)$ is regular and in (18) we have equality.

If $-j(x,t,\xi,\zeta,\cdot)$ is regular in the sense of Clarke, then we proceed similarly as above and we obtain that $-J(t,v,w,\cdot)$ is regular. Finally, we use the property $(-j)^0(x,t,\xi,\zeta,\eta;\bar\eta) = j^0(x,t,\xi,\zeta,\eta;-\bar\eta)$ for all ξ, ζ, η, $\bar\eta \in \mathbb{R}^d$ and we again get the equality in (18).

It remains to show that for $t \in (0,T)$ the multivalued mapping $\partial J(t,\cdot,\cdot,\cdot)$ is usc from $[L^2(\Gamma_C;\mathbb{R}^d)]^3$ to $L^2(\Gamma_C;\mathbb{R}^d)$ supplied with the weak topology. Since this mapping is locally relatively compact (i.e., for every $\bar w \in [L^2(\Gamma_C;\mathbb{R}^d)]^3$, there exists a neighborhood $\mathcal{O}_{\bar w}$ of $\bar w$ such that $\partial J(t,\mathcal{O}_{\bar w})$ is weakly compact subset of $L^2(\Gamma_C;\mathbb{R}^d)$), from Proposition 4.1.16 of [27], it is enough to show that $\operatorname{Graph}\partial J(t,\cdot,\cdot,\cdot)$ is closed in $[L^2(\Gamma_C;\mathbb{R}^d)]^3 \times (w - L^2(\Gamma_C;\mathbb{R}^d))$ topology.

Let $\{v_n\}$, $\{w_n\}$, $\{z_n\} \subset L^2(\Gamma_C;\mathbb{R}^d)$, $v_n \to v$, $w_n \to w$, and $z_n \to z$ in $L^2(\Gamma_C;\mathbb{R}^d)$, $\{z_n^*\} \subset L^2(\Gamma_C;\mathbb{R}^d)$, $z_n^* \to z^*$ weakly in $L^2(\Gamma_C;\mathbb{R}^d)$ and $z_n^* \in \partial J(t,v_n,w_n,z_n)$. The latter is equivalent to $\langle z_n^*,\bar v\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} \le J^0(t,v_n,w_n,z_n;\bar v)$ for all $\bar v \in L^2(\Gamma_C;\mathbb{R}^d)$. We may also assume, by passing to subsequences if necessary that $v_n(x) \to v(x)$, $w_n(x) \to w(x)$, and $z_n(x) \to z(x)$ in $\mathbb{R}^d$ for a.e. $x \in \Gamma_C$, $\|v_n(x)\|_{\mathbb{R}^d} \le v_0(x)$, $\|w_n(x)\|_{\mathbb{R}^d} \le w_0(x)$, and $\|z_n(x)\|_{\mathbb{R}^d} \le z_0(x)$ with $v_0, w_0, z_0 \in L^2(\Gamma_C)$. Let $\bar v \in L^2(\Gamma_C;\mathbb{R}^d)$. By the Fatou lemma, $H(j)_1$(ii), and equality (18), we have

$$\begin{aligned}
\limsup J^0\left(t,v_n,w_n,z_n;\bar v\right) &\le \limsup \int_{\Gamma_C} j^0\left(x,t,v_n(x),w_n(x),z_{nT}(x);\bar v_T(x)\right)\, d\Gamma(x) \\
&\le \int_{\Gamma_C} \limsup j^0\left(x,t,v_n(x),w_n(x),z_{nT}(x);\bar v_T(x)\right)\, d\Gamma(x) \\
&\le \int_{\Gamma_C} j^0\left(x,t,v(x),w(x),z_T(x);\bar v_T(x)\right)\, d\Gamma(x) = J^0\left(t,v,w,z;\bar v\right).
\end{aligned}$$

Hence we deduce

$$\langle z^*,\bar v\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} \le \limsup J^0\left(t,v_n,w_n,z_n;\bar v\right) \le J^0\left(t,v,w,z;\bar v\right)$$

for all $\bar v \in L^2(\Gamma_C;\mathbb{R}^d)$ which proves that $z^* \in \partial J(t,v,w,z)$. Thus $\operatorname{Graph}\partial J(t,\cdot,\cdot,\cdot)$ is closed in the aforementioned topology and so $\partial J(t,\cdot,\cdot,\cdot)$ is usc. The proof of the lemma is complete. ∎

Proof of Lemma 17. Proof of (a). Let $v \in \mathcal{V}$ and $z \in \mathcal{N}_1 v$. Hence $z(t) = \gamma^* w(t)$ and $w(t) \in \partial J(t,\gamma K(v(t)+u_1),\gamma(v(t)+u_1),\gamma(v(t)+u_1))$ for a.e. $t \in (0,T)$. Using $H(J)$(iii), we have

$$\begin{aligned}
\|w(t)\|_{L^2(\Gamma_C;\mathbb{R}^d)} &\le \tilde c\left(1+\|\gamma(v(t)+u_1)\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right) \le \tilde c\left(1+\|\gamma\|\|v(t)+u_1\|_Z\right) \\
&\le \tilde c\left(1+\beta\|\gamma\|\|v(t)+u_1\|\right) \le \tilde c\left(1+\beta\|\gamma\|\left(\|v(t)\|+\|u_1\|\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|z\|_{\mathcal{Z}^*} = \left(\int_0^T \|\gamma^* w(t)\|_{Z^*}^2\, dt\right)^{1/2} &\le \|\gamma^*\|\left(\int_0^T \|w(t)\|_{L^2(\Gamma_C;\mathbb{R}^d)}^2\, dt\right)^{1/2} \\
\le \bar c\,\|\gamma^*\|\left(1+T\|u_1\|+\beta\|\gamma\|\|v\|_{\mathcal{V}}\right),
\end{aligned}$$

which proves Property (a).

We now demonstrate Property (b). From the well-known properties of the Clarke subdifferential (see Proposition 2.1.2 of [29]) it follows that for every $v \in \mathcal{V}$ the set $\mathcal{N}_1 v$ is nonempty and convex in $\mathcal{Z}^*$. To show that $\mathcal{N}_1 v$ is weakly compact in $\mathcal{Z}^*$, we will prove that it is closed in $\mathcal{Z}^*$. Let $v \in \mathcal{V}$, $\{w_n\} \subset \mathcal{N}_1 v$, $w_n \to w$ in $\mathcal{Z}^*$. Passing to a subsequence if necessary, we suppose that $w_n(t) \to w(t)$ in Z^* for a.e. $t \in (0,T)$. Since $w_n(t) \in \gamma^*\,\partial J(t,\gamma K(v(t)+u_1)$, $\gamma(v(t)+u_1)$, $\gamma(v(t)+u_1))$ for a.e. $t \in (0,T)$ and the latter is a closed subset of Z^*, we get $w(t) \in \gamma^*\,\partial J(t,\gamma K(v(t)+u_1),\gamma(v(t)+u_1),\gamma(v(t)+u_1))$ for a.e. $t \in (0,T)$. Hence $w \in \mathcal{N}_1 v$. Thus the set $\mathcal{N}_1 v$ is closed in $\mathcal{Z}^*$ and convex, so it is also weakly closed in $\mathcal{Z}^*$. Since (by Property (a))

$\mathcal{N}_1 v$ is a bounded set in a reflexive Banach space $\mathcal{Z}^*$, we obtain that $\mathcal{N}_1 v$ is weakly compact in $\mathcal{Z}^*$.

Next, we will show (c). Let $v \in \mathcal{V}$ and $z \in \mathcal{N}_1 v$. So $z(t) = \gamma^* w(t)$ with $w(t) \in \partial J(t, \gamma K(v(t) + u_1), \gamma(v(t) + u_1), \gamma(v(t) + u_1))$ for a.e. $t \in (0,T)$. Using inequality $H(J)$(iv), we have

$$\begin{aligned} -\langle w(t), \gamma(v(t) + u_1)\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} &\le J^0\left(t, \gamma K(v(t) + u_1), \gamma(v(t) + u_1), \gamma(v(t) + u_1); -\gamma(v(t) + u_1)\right) \\ &\le \tilde{d}\left(1 + \|\gamma(v(t) + u_1)\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right) \le \tilde{d}\left(1 + \beta\|\gamma\|\|v(t) + u_1\|\right), \end{aligned}$$

for a.e. $t \in (0,T)$ with $\tilde{d} \ge 0$. On the other hand from $H(J)$(iii), we obtain

$$\begin{aligned} \langle\langle z, u_1\rangle\rangle_{\mathcal{Z}^*\times\mathcal{Z}} &\le \int_0^T \|z(t)\|_{Z^*}\|u_1\|_Z\, dt \le \beta\|u_1\| \int_0^T \|z(t)\|_{Z^*}\, dt \\ &\le \beta\tilde{c}\|u_1\|\|\gamma\| \int_0^T \left(1 + \|\gamma(v(t) + u_1)\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right) dt \\ &\le \beta\tilde{c}\|u_1\|\|\gamma\| \left(T + \beta T\|\gamma\|\|u_1\| + \beta\sqrt{T}\|\gamma\|\|v\|_{\mathcal{V}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \langle\langle z, w\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle\langle z, w\rangle\rangle_{\mathcal{Z}^*\times\mathcal{Z}} &= \int_0^T \langle\gamma^* w(t), v(t) + u_1\rangle_{Z^*\times Z}\, dt - \int_0^T \langle z(t), u_1\rangle_{Z^*\times Z}\, dt \\ &\ge -\tilde{d}\int_0^T \left(1 + \beta\|\gamma\|\|v(t)\|\right) dt - \tilde{d}T\left(1 + \beta\|\gamma\|\|u_1\|\right) \\ &\quad - \beta\tilde{c}\|u_1\|\|\gamma\|\left(T + \beta T\|\gamma\|\|u_1\|\right) - \beta^2\tilde{c}\sqrt{T}\|u_1\|\|\gamma\|^2\|v\|_{\mathcal{V}} \ge -\tilde{c}_1\|v\|_{\mathcal{V}} - \tilde{c}_2 \end{aligned}$$

with $\tilde{c}_1$, $\tilde{c}_2 > 0$, and Property (c) follows.

Finally, let $z_n \in \mathcal{N}_1 v_n$ with v_n, $v \in \mathcal{V}$, $v_n \to v$ in $\mathcal{Z}$ and z_n, $z \in \mathcal{Z}^*$ and $z_n \to z$ weakly in $\mathcal{Z}^*$. By passing to a subsequence if necessary, we may assume $v_n(t) \to v(t)$ in Z for a.e. $t \in (0,T)$. This gives

$$\gamma(v_n(t) + u_1) \to \gamma(v(t) + u_1), \qquad \text{in } L^2\left(\Gamma_C;\mathbb{R}^d\right), \quad \text{for a.e. } t \in (0,T). \tag{53}$$

Since

$$\begin{aligned} &\|K(v_n + u_1) - K(v + u_1)\|_{\mathcal{Z}}^2 \\ &\quad = \int_0^T \left\| \int_0^t v_n(s)\, ds + u_1 t + u_0 - \int_0^t v(s)\, ds - u_1 t - u_0 \right\|_Z^2 dt \le T\|v_n - v\|_{\mathcal{Z}}^2 \to 0, \end{aligned}$$

and γ is a continuous operator, we get

$$\gamma K(v_n(t) + u_1) \to \gamma K(v(t) + u_1), \qquad \text{in } L^2\left(\Gamma_C;\mathbb{R}^d\right), \quad \text{for a.e. } t \in (0,T). \tag{54}$$

Next, we have

$$z_n(t) = \gamma^*\eta_n(t), \qquad \text{for a.e. } t \in (0,T), \tag{55}$$

with

$$\eta_n(t) \in \partial J(t, \gamma K(v_n(t) + u_1), \gamma(v_n(t) + u_1), \gamma(v_n(t) + u_1)), \qquad \text{for a.e. } t \in (0,T). \tag{56}$$

From Property (a) of this lemma we deduce that the sequence $\{\eta_n\}$ is bounded in $L^2(0,T; L^2(\Gamma_C;\mathbb{R}^d))$ and so we may suppose that

$$\eta_n \to \eta, \qquad \text{weakly in } L^2\left(0,T; L^2\left(\Gamma_C;\mathbb{R}^d\right)\right), \tag{57}$$

for some $\eta \in L^2(0,T; L^2(\Gamma_C;\mathbb{R}^d))$. Using (57) and the convergence $z_n \to z$ weakly in $\mathcal{Z}^*$, from (55) we obtain $z(t) = \gamma^*\eta(t)$ for a.e. $t \in (0,T)$. From (53), (54), (57) and since $\partial J(t,\cdot,\cdot,\cdot)$ is usc with closed and convex values, we apply the convergence theorem of Aubin and Cellina [35, p. 60]) to inclusion (56). We obtain $\eta(t) \in \partial J(t, \gamma K(v(t) + u_1), \gamma(v(t) + u_1), \gamma(v(t) + u_1))$ for a.e. $t \in (0,T)$, which implies $z \in \mathcal{N}_1 v$. The proof of the lemma is complete. ∎

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